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Spectral discretizations of the Darcy's equations with non standard boundary conditions

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Abstract

This paper is devoted to the approximation of a nonstandard Darcy problem, which modelizes the ow in porous media, by spectral methods: the pressure is assigned on a part of the boundary. We propose two variational formulations, as well as three spectral discretizations. The second discretization improves the approximation of the divergence-free condition, but the error estimate on the pressure is not optimal, while the third one leads to optimal error estimate with a divergence-free discrete solution, which is important for some applications. Next, their numerical analysis is performed in detail and we present some numerical experiments which conrm the interest of the third discretization.

Keywords: Key words: Spectral methods; Darcy equations; boundary conditions; numerical results.

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Introduction

We consider the following Darcy problem:

$$\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{1.1}$$
$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1.2}$$
$$\mathbf{u} \,.\, \mathbf{n} = U_0 \qquad \text{on } \Gamma^1, \tag{1.3}$$
$$p = \varphi \qquad \text{on } \Gamma^2, \tag{1.4}$$

where Ω is the plane square $]-1,1[^2$ and $\mathbf{n} = (n_1, n_2)$ is the exterior unit normal to the boundary $\Gamma = \partial\Omega$. The boundary Γ is divided into two parts: the horizontal portion $\Gamma^1 = \{(x,y) | -1 < x < 1, y = \pm 1\}$ and the vertical portion $\Gamma^2 = \{(x,y) | x = \pm 1, -1 < y < 1\}$. As we can see, the boundary condition on Γ^2 are nonstandard, since we prescribe the value of the pressure on Γ^2. On the contrary, we have a classical condition on the portion Γ^1. These conditions are described in the following figure.

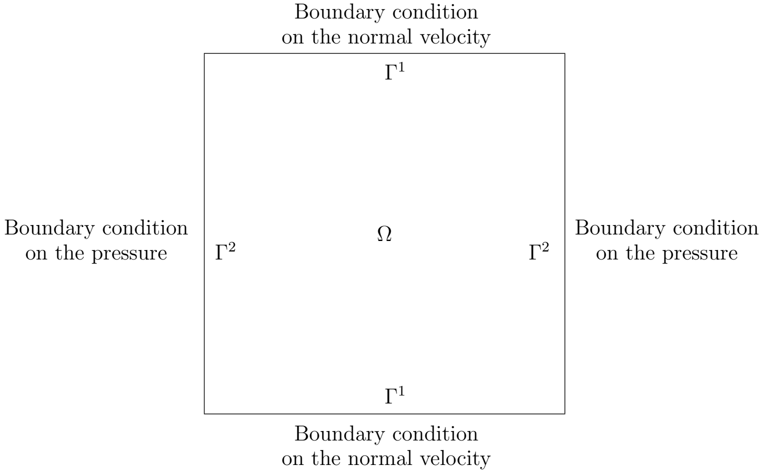

Figure 1.1

The equations of the Darcy problem not only modelize the flow in porous media, but also appear in the projection techniques for the solution of Navier-Stokes equations (see [10] and [15]). The nonstandard boundary conditions, where the pressure is assigned on a part of the boundary, have a physical meaning: typically the portion Γ^1 corresponds to rigid walls, whereas the entry or exit of the fluids takes place through Γ^2. The spectral discretization with this type of boundary conditions was only studied within the framework of the Stokes problem (see [6], [4] and [5]), while Azaïez, Bernardi and Grundmann proposed in [2] the spectral discretization of the standard Darcy problem, where the normal velocity is assigned on the boundary.

This paper is devoted to the spectral discretization of the nonstandard Darcy problem. First, we give two variational formulations. Each one leads us to well-posed problems. Second, we study the regularity of the solution by using a mixed problem of Dirichlet-Neumann for the Laplace operator. Next, from the first variational formulation, we derive a first spectral discretization, which is simple, but, in order to improve the approximation of the divergence-free condition, we study a second spectral discretization, where the inf-sup condition is obtained with more difficulty and where the error estimate on the pressure is not optimal. Finally, the second variational formulation yields a third spectral discretization, which leads us to optimal error estimate and a divergence-free discrete solution.

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An outline of this paper is as follows. The two continuous variational problems, as well as the regularity of the solution, are studied in Section 2. Section 3 is devoted to the analysis of three spectral approximations of this problem in the case of homogeneous boundary conditions. In Section 4, we present the algorithms that are used to solve the first and third discretizations, together with some numerical experiments.

Statement of the problem and notation

In order to set this problem into adequate spaces, recall the definition of the following standard Sobolev spaces (cf. J. Nečas [13]). For any multi-index $k = (k_1, k_2)$ with $k_i \geq 0$, set $|k| = k_1 + k_2$ and denote

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2}}.$$

Then for any integer $m \geq 0$ and any plane domain Ω whose boundary is Lipschitz-continuous(cf. Grisvard [12]), we define:

$$H^m(\Omega) = \{v \in L^2(\Omega); \partial^k v \in L^2(\Omega) \text{ for } 1 \leq |k| \leq m\},$$

equipped with the seminorm

$$|v|_{H^m(\Omega)} = (\sum_{|k|=m} \int_\Omega |\partial^k v|^2 \, d\mathbf{x})^{\frac{1}{2}},$$

and norm(for which it is an Hilbert space)

$$\|v\|_{H^m(\Omega)} = (\sum_{|k|=0}^{m} \sum_{k} \|\partial^k v\|^2_{L^2(\Omega)})^{1/2}.$$

For extensions of this definition to non-integral values of m (see [11,12]), let s a real number such that $s = m + \sigma$ with $m \in \mathbb{N}$ and $0 < \sigma < 1$. We denote by $H^s(\Omega)$ the space of all distributions u defined in Ω such that $u \in H^m(\Omega)$ and, $\forall |\alpha| = m$,

$$\int_\Omega \int_\Omega \frac{(\partial^\alpha u(x) - \partial^\alpha u(y))^2}{\|x - y\|^{2+2\sigma}} \, dxdy < +\infty.$$

It can be shown that $H^s(\Omega)$ is a Hilbert space for the scalar product

$$(u, v)_{s,\Omega} = (u, v)_{m,\Omega} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{(\partial^\alpha u(x) - \partial^\alpha u(y))(\partial^\alpha v(x) - \partial^\alpha v(y))}{\|x - y\|^{2+2\sigma}} \, dxdy. \tag{2.1}$$

Let Γ' be an open part of the boundary $\partial\Omega$ of class $C^{m-1,1}$ and $T_1^{\Gamma'}$ the mapping $v \mapsto v_{|\Gamma'}$ defined on $H^m(\Omega)$. We denote by $H^{m-\frac{1}{2}}(\Gamma')$ (see [7,12]) the space $T_1^{\Gamma'}(H^m(\Omega))$ which is equipped with the norm:

$$\|\varphi\|_{H^{m-\frac{1}{2}}(\Gamma')} = \inf\{\|v\|_{H^m(\Omega)}, \; v \in H^m(\Omega) \text{ and } v_{|\Gamma'} = \varphi\}. \tag{2.2}$$

In this text, we shall use the spaces $H^{1/2}(\Gamma')$ and $H^{3/2}(\Gamma')$ corresponding to $m = 1$ and 2.

Let us define the space $H_{00}^{1/2}(\Gamma') = \{v_{|\Gamma'}, \; v \in H^1(\Omega), \; \forall \mathbf{x} \in \partial\Omega \setminus \Gamma', \; v_{|\partial\Omega}(\mathbf{x}) = 0\}$. We shall also be interested in the dual space of $H_{00}^{1/2}(\Gamma')$,

$$H^{-1/2}(\Gamma') = (H_{00}^{1/2}(\Gamma'))'. \tag{2.3}$$

We shall use the Hilbert space $H(\text{div}\,;\Omega) = \{\mathbf{v} \in L^2(\Omega)^2 : \; \text{div}\,\mathbf{v} \in L^2(\Omega)\}$, equipped with the norm

$$\|\mathbf{v}\|_{H(\mathrm{div}\,;\Omega)} = ((\|\mathbf{v}\|_{L^2(\Omega)})^2 + (\|\mathrm{div}\,\mathbf{v}\|_{L^2(\Omega)})^2)^{\frac{1}{2}}. \tag{2.4}$$

For vanishing boundary values, we define:

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : \; v_{|\partial\Omega} = 0\}.$$

For $\Lambda =]-1, 1[$, we denote the norm in $L^2(\Lambda)$ by $\|v\|_{0,\Lambda} = (\int_{-1}^{1} (v(x))^2 \, dx)^{\frac{1}{2}}$, the semi-norm in $H^1(\Lambda)$ by $|v|_{1,\Lambda} = (\int_{-1}^{1} (v'(x))^2 \, dx)^{\frac{1}{2}}$ and the norm in $H^1(\Omega)$ by $\|v\|_{1,\Lambda} = (\int_{-1}^{1} ((v(x))^2 + (v'(x))^2) \, dx)^{\frac{1}{2}}$.

We note $\mathbf{x} = (x, y)$ the generic point of the square Ω and we call Γ_I, Γ_{II}, Γ_{III} and Γ_{IV} the edges of Ω, starting from west and turning counterclockwise. For each J, $J = I, II, III, IV$, the extremities of the edge Γ_J are $\mathbf{a_{J-1}}$ and $\mathbf{a_J}$, with the convention $\mathbf{a_0} = \mathbf{a_{IV}}$, the exterior unit normal vector to Γ_J is denoted by $\mathbf{n_J}$ and the counterclockwise unit tangent vector is $\boldsymbol{\tau}_\mathbf{J}$. Figure 1.2 below presents this notation.

For any domain Δ in $\mathbb{R}$ or $\mathbb{R}^2$ and for any nonnegative integer n, $\mathbb{P}_n(\Delta)$ stands for the space of all polynomials on Δ with degree $\leq n$ with respect to each variable. We also use the notation $\mathbb{P}_n^0(\Delta)$ for the subspace $\mathbb{P}_n(\Delta) \cap H_0^1(\Delta)$. For $\Lambda =]-1, 1[$, the family $(L_n)_n$ of Legendre polynomials is a basis of the spaces $\mathbb{P}(\Lambda)$ of polynomials on Λ (we refer to [7, Chap. I] for the properties of the orthogonal polynomials). These polynomials are orthogonal to each other in $L^2(\Lambda)$ and are caracterized as follows: for any integer $n \geq 0$, the polynomial L_n is of degree n and satisfies $L_n(1) = 1$. Let us recall some properties that we need. The family $(L_n)_n$ is given by the recursion relation:

$$\begin{cases} L_0 = 1, \; L_1(\zeta) = \zeta, \\ (n+1)L_{n+1}(\zeta) = (2n+1)\zeta L_n(\zeta) - nL_{n-1}(\zeta), \qquad n \geq 1. \end{cases} \tag{2.5}$$

Each polynomial is a solution of the differential equation

$$((1-\zeta^2)L_n')' + n(n+1)L_n = 0, \qquad n \geq 0, \tag{2.6}$$

and its norm is given by

$$\|L_n\|_{0,\Lambda}^2 = \frac{2}{2n+1}, \qquad n \geq 0. \tag{2.7}$$

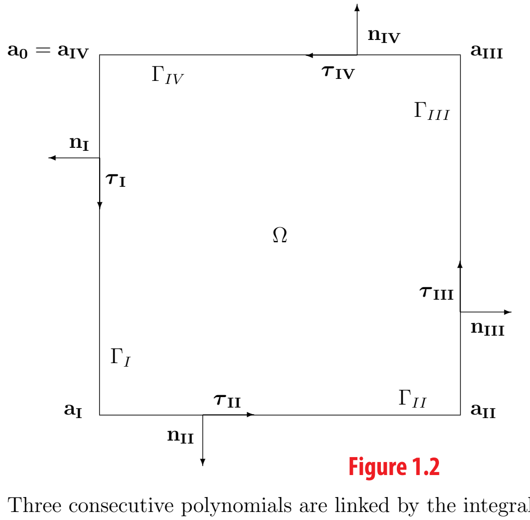

Figure 1.2

Three consecutive polynomials are linked by the integral equation

$$\int_{-1}^{\zeta} L_n(\xi)\, d\xi = \frac{1}{2n+1}(L_{n+1}(\zeta) - L_{n-1}(\zeta)), \qquad n \geq 1. \tag{2.8}$$

From (2.6) and integration by parts, we derive

$$L'_n(1) = \frac{n(n+1)}{2}, \ \|L'_n\|^2_{0,\Lambda} = n(n+1), \tag{2.9}$$

$$\forall \varphi_n \in \mathbb{P}_n(\Lambda), \ |\varphi_n|_{1,\Lambda} \leq \sqrt{3}n^2 \|\varphi_n\|_{0,\Lambda}. \tag{2.10}$$

Next, let $N \geq 2$ be a fixed integer. We denote by ξ_j, $0 \leq j \leq N$, the zeros of the polynomial $(1-\zeta^2)L'_N(\zeta)$ in increasing order. We recall (see [7, Chap. I]) that there exist positive weights ρ_j, $0 \leq j \leq N$, such that the following equality, called the Gauss-Lobatto formula, holds

$$\forall \Phi \in \mathbb{P}_{2N-1}(\Lambda), \int_{-1}^{1} \Phi(\zeta)\, d\zeta = \sum_{j=0}^{N} \Phi(\xi_j)\rho_j. \tag{2.11}$$

Moreover, it follows from the identity (see [7, Chap. III])

$$\sum_{j=0}^{N} L_N(\xi_j)^2 \rho_j = (2 + \frac{1}{N}) \|L_N\|^2_{0,\Lambda}, \tag{2.12}$$

that the bilinear form: $(u,v) \rightarrow \sum_{j=1}^{N} u(\xi_j)v(\xi_j)\rho_j$ is a scalar product on $\mathbb{P}_N(\Lambda)$, since we have

$$\forall v \in \mathbb{P}_N(\Lambda), \ \|v\|^2_{0,\Lambda} \leq \sum_{j=0}^{N} v(\xi_j)^2 \rho_j \leq 3\|v\|^2_{0,\Lambda}. \tag{2.13}$$

THE CONTINUOUS PROBLEM

First variational formulation

We define the subspace $H^1(\Omega;\Gamma') = \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma'\}$ of $H^1(\Omega)$ and we introduce the space:

$$M = H^1(\Omega;\Gamma^2). \tag{3.1}$$

In the same way as in [1], we consider the following equivalent variational formulation of the problem (1.1)-(1.4):

Find $\mathbf{u}$ in $L^2(\Omega)^2$ and p in $H^1(\Omega)$ such that $p - \varphi$ belongs to M and that

$$\forall \mathbf{v} \in L^2(\Omega)^2, \qquad a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \int_\Omega \mathbf{f}(\mathbf{x})\,.\,\mathbf{v}(\mathbf{x})\, d\mathbf{x}, \tag{3.2}$$

$$\forall q \in M, \qquad b(\mathbf{u},q) = < U_0, q >_{\Gamma^1}, \tag{3.3}$$

where the bilinear forms a and b are defined by

$$\forall (\mathbf{v},\mathbf{w}) \in (L^2(\Omega)^2)^2, \qquad a(\mathbf{v},\mathbf{w}) = \int_\Omega \mathbf{v}(\mathbf{x})\,.\,\mathbf{w}(\mathbf{x})\, d\mathbf{x}, \tag{3.4}$$

$$\forall \mathbf{v} \in L^2(\Omega)^2, \ \forall q \in H^1(\Omega), \qquad b(\mathbf{v},q) = \int_\Omega \mathbf{v}(\mathbf{x})\,.\,\nabla q(\mathbf{x})\, d\mathbf{x}. \tag{3.5}$$

Theorem 3.1 *Let $\mathbf{f}$ be in $L^2(\Omega)^2$, U_0 in $H^{-1/2}(\Gamma^1)$ and φ in $H^{1/2}(\Gamma^2)$, where $H^{-1/2}(\Gamma^1)$ and $H^{1/2}(\Gamma^2)$ are defined respectively in (2.3) and (2.2). Then problem (3.2), (3.3) has a unique solution satisfying*

$$\|\mathbf{u}\|_{L^2(\Omega)^2} + \|p\|_{H^1(\Omega)} \leq C(\|\mathbf{f}\|_{L^2(\Omega)} + \|U_0\|_{-1/2,\Gamma^1} + \|\varphi\|_{1/2,\Gamma^2}). \tag{3.6}$$

Proof. First, let us define $\tilde{\varphi}$ belonging to $H^{1/2}(\Gamma)$ such that $\tilde{\varphi}_{|\Gamma^2} = \varphi$ and $\|\tilde{\varphi}\|_{1/2,\Gamma} \leq c\|\varphi\|_{1/2,\Gamma^2}$. To this end, we must extend φ to a function belonging to $H^{1/2}(\Gamma)$. Let μ be a function defined in $[0,2]$ by

$$\mu(t) = 1 - t, \ \text{for } 0 \leq t \leq 1 \text{ and } \mu(t) = 0, \ \text{for } 1 \leq t \leq 2.$$

We define $\tilde{\varphi}_{|\Gamma_{II}}$ by

$$\tilde{\varphi}(\mathbf{a}_I + t\boldsymbol{\tau}_{II}) = \mu(t)\varphi(\mathbf{a}_0 + (2-t)\boldsymbol{\tau}_I) + \mu(2-t)\varphi(\mathbf{a}_{II} + (2-t)\boldsymbol{\tau}_{III})$$

and $\tilde{\varphi}_{|\Gamma_{IV}}$ by

$$\tilde{\varphi}(\mathbf{a}_O - t\boldsymbol{\tau}_{IV}) = \mu(t)\varphi(\mathbf{a}_0 + t\boldsymbol{\tau}_I) + \mu(2-t)\varphi(\mathbf{a}_{II} + t\boldsymbol{\tau}_{III}).$$

Then we have $\|\tilde{\varphi}\|_{1/2,\Gamma} \leq C\|\varphi\|_{1/2,\Gamma^2}$. Next, let Φ in $H^1(\Omega)$ such that $\Phi_{|\Gamma} = \tilde{\varphi}$. Finally, we obtain a function Φ verifying

$$\Phi_{|\Gamma^2} = \varphi \text{ and } \|\Phi\|_{H^1(\Omega)} \leq C\|\varphi\|_{1/2,\Gamma^2}. \tag{3.7}$$

Second, let us extend U_0 to a function belonging to $H^{-1/2}(\Gamma)$. We set $\tilde{U}_{0|\Gamma^1} = U_0$ and $\tilde{U}_{0|\Gamma^2} = -\frac{1}{2} < U_0, 1 >_{\Gamma^1}$. Then, we have $\|\tilde{U}_0\|_{-1/2,\Gamma} \leq C\|U_0\|_{-1/2,\Gamma^1}$ and $< \tilde{U}_0, 1 >_\Gamma = 0$. Next, we define Neumann's Problem:

$$\begin{cases} -\Delta\psi = 0 & \text{in } \Omega \\ \dfrac{\partial\psi}{\partial n} = \tilde{U}_0 & \text{on } \Gamma \end{cases}$$

and we set $\mathbf{u}_0 = \nabla\psi$. Applying Proposition 1.2 of [11, page 14], we derive that ψ belongs to $H^1(\Omega)$, which implies that $\mathbf{u}_0$ belongs to $H(\mathrm{div}\,;\Omega)$ with

$$\|\mathbf{u}_0\|_{H(\mathrm{div}\,;\Omega)} = |\psi|_{H^1(\Omega)} \leq C\|\tilde{U}_0\|_{-1/2,\Gamma},$$

since $\mathrm{div}\,\mathbf{u}_o = \Delta\psi = 0$. Finally, $\mathbf{u}_0$ verifies

$$\mathbf{u}_0 \,.\, \mathbf{n}_{|\Gamma^1} = U_0,\ \mathrm{div}\,\mathbf{u}_0 = 0 \text{ in } \Omega \text{ and } \|\mathbf{u}_0\|_{H(\mathrm{div}\,;\Omega)} \leq C\|U_0\|_{-1/2,\Gamma^1}. \tag{3.8}$$

Now, let us split p as: $p = \Phi + \tilde{p}$ with $\tilde{p}$ in M and $\mathbf{u}$ as: $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$. Then, we can write the problem (3.2),(3.3) as

$$\forall \mathbf{v} \in L^2(\Omega)^2, \qquad a(\tilde{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \tilde{p}) = \int_\Omega (\mathbf{f}(\mathbf{x}) - \nabla\Phi(\mathbf{x}) - \mathbf{u}_0(\mathbf{x}))\,.\,\mathbf{v}(\mathbf{x})\,d\mathbf{x}. \tag{3.9}$$

$$\forall q \in M, \qquad b(\tilde{\mathbf{u}}, q) = 0. \tag{3.10}$$

Since the right-hand side of (3.9) defines a continuous form on $L^2(\Omega)^2$ and since the properties of continuity and ellipticity are obvious we have only to check the following inf-sup condition on the form b (see [11, pages 58,59]):

$$\inf_{q\in M} \sup_{\mathbf{v}\in L^2(\Omega)^2} \frac{b(\mathbf{v}, q)}{\|v\|_{L^2(\Omega)^2}} \geq \beta \iff \forall q \in M, \qquad \sup_{\mathbf{v}\in L^2(\Omega)^2} \frac{b(\mathbf{v}, q)}{\|v\|_{L^2(\Omega)^2}} \geq \beta\|q\|_{H^1(\Omega)}, \tag{3.11}$$

with a positive constant β. This "inf-sup condition" was introduced independently by Babuska [3] and Brezzi [9]. We can verify this condition by taking $\mathbf{v} = \nabla q$. Indeed, we have

$$\sup_{\mathbf{v}\in L^2(\Omega)^2} \frac{b(\mathbf{v}, q)}{\|v\|_{L^2(\Omega)^2}} \geq \frac{b(\nabla q, q)}{\|\nabla q\|_{L^2(\Omega)^2}} = |q|_{H^1(\Omega)}$$

and, since $q_{|\Gamma^2} = 0$, using a generalization of Poincaré inequality (see [11, Chap. I, page 40]) yields $\|q\|_{L^2(\Omega)} \leq \mathcal{P}|q|_{H^1(\Omega)}$, which implies

$$\|q\|_{H^1(\Omega)} \leq \sqrt{(\mathcal{P})^2 + 1)}|q|_{H^1(\Omega)}.$$

Thus, the "inf-sup condition" is verified with the positive constant $\beta = \dfrac{1}{\sqrt{(\mathcal{P})^2 + 1)}}$. Hence, applying Theorem 2.3 [7, pages 116,117], the theorem follows. ♢

Second variational formulation

We introduce the space:

$$X = \{\mathbf{v} \in L^2(\Omega)^2;\ \operatorname{div} \mathbf{v} \in L^2(\Omega) \text{ and } \mathbf{v} \,.\, \mathbf{n}_{|\Gamma^1} = 0\}. \tag{3.12}$$

In an analogous way as in [1], we consider the following equivalent variational formulation of the problem (1.1)-(1.4):

Find $\mathbf{u}$ in X and p in $L^2(\Omega)$ such that $\mathbf{u} - \mathbf{u}_0$ belongs to X, where $\mathbf{u}_0$ is the function previously constructed that verifies (3.8), and that

$$\forall \mathbf{v} \in X, \qquad a(\mathbf{u}, \mathbf{v}) + b^*(\mathbf{v}, p) = \int_\Omega \mathbf{f}(\mathbf{x}) \,.\, \mathbf{v}(\mathbf{x}) \, d\mathbf{x} - < \varphi, \mathbf{v} \,.\, \mathbf{n} >_{\Gamma^2}, \tag{3.13}$$

$$\forall q \in L^2(\Omega), \qquad b^*(\mathbf{u}, q) = 0, \tag{3.14}$$

where the bilinear form b^* is defined by

$$\forall \mathbf{v} \in H(\operatorname{div}; \Omega),\ \forall q \in L^2(\Omega), \qquad b^*(\mathbf{v}, q) = -\int_\Omega \operatorname{div} \mathbf{v}(\mathbf{x}) \,.\, q(\mathbf{x}) \, d\mathbf{x}. \tag{3.15}$$

In the same way as previously, we split $\mathbf{u}$ as: $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}$. Then, we can write the problem (3.13),(3.14) as

$$\forall \mathbf{v} \in X, \qquad a(\tilde{\mathbf{u}}, \mathbf{v}) + b^*(\mathbf{v}, p) = \int_\Omega (\mathbf{f}(\mathbf{x}) - \mathbf{u}_0(\mathbf{x})) \,.\, \mathbf{v}(\mathbf{x}) \, d\mathbf{x} - < \varphi, \mathbf{v} \,.\, \mathbf{n} >_{\Gamma^2}. \tag{3.16}$$

$$\forall q \in L^2(\Omega), \qquad b^*(\tilde{\mathbf{u}}, q) = 0. \tag{3.17}$$

Since the right-hand side of (3.16) defines a continuous form on X and since the properties of continuity and ellipticity are obvious, we have only to check the following inf-sup condition on the form b^*:

$$\forall q \in L^2(\Omega), \qquad \sup_{\mathbf{v} \in X} \frac{b^*(\mathbf{v}, q)}{\|\mathbf{v}\|_{H(\operatorname{div}; \Omega)}} \geq \beta^* \|q\|_{L^2(\Omega)}, \tag{3.18}$$

with a positive constant β^*. Let us note that $q_0 = q - \frac{1}{|\Omega|} \int_\Omega q(\mathbf{x}) \, d\mathbf{x}$ belongs to $L^2_0(\Omega)$ and, owing to a classic result (see [11, Chap. I]), there exists $\mathbf{v}_0$ in $H^1_0(\Omega)^2$, such that

$$\operatorname{div} \mathbf{v}_0 = -q_0 \qquad \text{and} \qquad \|\mathbf{v}_0\|_{H^1(\Omega)^2} \leq c \|q_0\|_{L^2(\Omega)}.$$

Then, we set

$$\tilde{\mathbf{v}} = \mathbf{v}_0 + \mathbf{v}_1 \qquad \text{with} \qquad \mathbf{v}_1(x, y) = (-(\frac{1}{|\Omega|} \int_\Omega q(\mathbf{x}) \, d\mathbf{x}) x, 0),\ -1 \leq x, y \leq 1.$$

We can verify that $\tilde{\mathbf{v}}$ belongs to X with

$$\operatorname{div} \tilde{\mathbf{v}} = -q \qquad \text{and} \qquad \|\tilde{\mathbf{v}}\|_{H(\operatorname{div}; \Omega)} \leq C \|\tilde{\mathbf{v}}\|_{H^1(\Omega)^2} \leq C' \|q\|_{L^2(\Omega)},$$

since $| \int_\Omega q(\mathbf{x}) \, d\mathbf{x}| \leq \sqrt{|\Omega|} \, \|q\|_{L^2(\Omega}$. Then, we have

$$\sup_{\mathbf{v} \in X} \frac{b^*(\mathbf{v}, q)}{\|\mathbf{v}\|_{H(\operatorname{div}; \Omega)}} \geq \frac{b^*(\tilde{\mathbf{v}}, q)}{\|\tilde{\mathbf{v}}\|_{H(\operatorname{div}; \Omega)}} \geq \frac{1}{C'} \|q\|_{L^2(\Omega)}.$$

Hence, we derive the inf-sup condition and we obtain the following result.

Theorem 3.2 *Let* $\mathbf{f}$ *be in* $L^2(\Omega)^2$, U_0 *in* $H^{-1/2}(\Gamma^1)$ *and* φ *in* $H^{1/2}(\Gamma^2)$, *where* $H^{-1/2}(\Gamma^1)$ *and* $H^{1/2}(\Gamma^2)$ *are defined respectively in (2.3) and (2.2). Then problem (3.13), (3.14) has a unique solution satisfying*

$$\|\mathbf{u}\|_{H(\operatorname{div}, \Omega)} + \|p\|_{L^2(\Omega)} \leq C(\|\mathbf{f}\|_{L^2(\Omega)} + \|U_0\|_{-1/2, \Gamma^1} + \|\varphi\|_{1/2, \Gamma^2}). \tag{3.19}$$

Regularity results

When the data $\mathbf{f}$ is in $H(\mathrm{div}\,;\Omega)$, taking the divergence of the first equation of the problem (1.1)-(1.4) and owing to the other equations, we obtain a mixed problem of Dirichlet-Neumann for the Laplace operator:

$$\begin{cases} \Delta p = \mathrm{div}\,\mathbf{f} & \text{in } \Omega \\ p = \varphi & \text{on } \Gamma^2 \\ \dfrac{\partial p}{\partial n} = \mathbf{f}\,.\,\mathbf{n} - U_0 & \text{on } \Gamma^1. \end{cases} \tag{3.20}$$

We suppose that $\mathbf{f}$ is in $H^1(\Omega)^2$, φ is in $H^{3/2}(\Gamma^2)$ and U_0 is in $H^{1/2}(\Gamma^1)$. In addition, we assume matching conditions at the vertices of Γ (see [7, Chap I]):

$$\begin{aligned} \int_0^2 |\frac{d\varphi}{d\boldsymbol{\tau}_J}(\mathbf{a}_J - t\boldsymbol{\tau}_J) - (\mathbf{f}\,.\,\mathbf{n} - U_0)(\mathbf{a}_J + t\boldsymbol{\tau}_{J+1})|^2 \frac{dt}{t} < +\infty,\ J = I, III \\ \int_0^2 |(\mathbf{f}\,.\,\mathbf{n} - U_0)(\mathbf{a}_J - t\boldsymbol{\tau}_J) + \frac{d\varphi}{d\boldsymbol{\tau}_{J+1}}(\mathbf{a}_J + t\boldsymbol{\tau}_{J+1})|^2 \frac{dt}{t} < +\infty,\ J = II, IV. \end{aligned} \tag{3.21}$$

Theorem 3.3 *For any data $\mathbf{f}$ in $H^1(\Omega)^2$, φ in $H^{3/2}(\Gamma^2)$ and U_0 in $H^{1/2}(\Gamma^1)$, where $H^{3/2}(\Gamma^2)$ and $H^{1/2}(\Gamma^1)$ are defined in (2.2) , verifying the matching conditions (3.21), the solution $(\mathbf{u}, p)$ of the problem (1.1)-(1.4) belongs to $H^1(\Omega)^2 \times H^2(\Omega)$.*

Proof. Owing to matching conditions (3.21), there exists p_0 in $H^2(\Omega)$ such that $p_{0|\Gamma^2} = \varphi$ and $(\frac{\partial p_0}{\partial n})_{|\Gamma^1} = \mathbf{f}\,.\,\mathbf{n} - U_0$. Let us set $\tilde{p} = p - p_0$. The problem (3.20) is equivalent to the following problem: find $\tilde{p}$ in $H^1(\Omega;\Gamma^2)$ such that

$$\forall q \in H^1(\Omega;\Gamma^2),\ a(\nabla\tilde{p}, \nabla q) = \int_\Omega (\mathrm{div}\,\mathbf{f} + \Delta p_0)(\mathbf{x}) q(\mathbf{x})\, d\mathbf{x}.$$

Since the boundary between Γ^1 and Γ^2 is the set of vertices of Γ, the regularity of the data implies that this homogeneous mixed problem of Dirichlet-Neumann for the Laplace operator has a solution $\tilde{p}$ in $H^2(\Omega)$ (see [12]). Hence, we derive the regularity of p and $\mathbf{u}$. $\diamondsuit$

Remark 3.4 *If $(\mathbf{f}\,.\,\mathbf{n} - U_0)$ is Lipschitz-continuous on $\overline{\Gamma_J}$ or belongs to $H^1(\Gamma_J)$, $J = II, IV$ and if φ belongs to $C^{1,1}(\overline{\Gamma_J})$ or to $H^2(\Gamma_J)$, $J = I, III$, the matching conditions (3.21) are equivalent to simpler conditions:*

$$\frac{d\varphi}{d\boldsymbol{\tau}_J}(\mathbf{a}_J) = (\mathbf{f}\,.\,\mathbf{n} - U_0)(\mathbf{a}_J),\ J = I, III \text{ and } -\frac{d\varphi}{d\boldsymbol{\tau}_{J+1}}(\mathbf{a}_J) = (\mathbf{f}\,.\,\mathbf{n} - U_0)(\mathbf{a}_J),\ J = II, IV.$$

SPECTRAL DISCRETIZATION

First spectral discretization

We define the discrete scalar product by

$$(u, v)_N = \sum_{i=0}^{N}\sum_{j=0}^{N} u(\xi_i, \xi_j) v(\xi_i, \xi_j) \rho_i \rho_j. \tag{4.1}$$

and we denote by $\mathcal{I}_N$ the Lagrange interpolation operator at the points (ξ_i, ξ_j), $0 \le i, j \le N$ in $\mathbb{P}_N(\Omega)$. We set

$$X_N = \mathbb{P}_N(\Omega)^2 \quad \text{or} \quad X_N = (\mathbb{P}_{N-1}(\Lambda) \otimes \mathbb{P}_N(\Lambda)) \times (\mathbb{P}_N(\Lambda) \otimes \mathbb{P}_{N-1}(\Lambda)). \tag{4.2}$$

We assume that the data $\mathbf{f}$ belongs to $C^0(\overline{\Omega})^2$ and, for the sake of simplicity, that $\mathbf{u}$ and p satisfy homogeneous boundary conditions, that is to say, we set $\varphi = 0$, $U_0 = 0$ in (1.1)-(1.4). From the variational formulation (3.2)-(3.3), we derive the following discrete problem:

Find $\mathbf{u}_N$ in X_N and p_N in $\mathbb{P}_N(\Omega) \cap M$, where M is defined by (3.1), such that

$$\forall \mathbf{v}_N \in X_N, \qquad (\mathbf{u}_N, \mathbf{v}_N)_N + b_N(\mathbf{v}_N, p_N) = (\mathbf{f}, \mathbf{v}_N)_N, \tag{4.3}$$
$$\forall q_N \in \mathbb{P}_N(\Omega) \cap M, \qquad b_N(\mathbf{u}_N, q_N) = 0, \tag{4.4}$$

where the form b_N is defined by

$$\forall \mathbf{v}_N \in \mathbb{P}_N(\Omega)^2,\ \forall q_N \in \mathbb{P}_N(\Omega),\ b_N(\mathbf{v}_N, q_N) = (\mathbf{v}_N, \nabla q_N)_N. \tag{4.5}$$

We have a classical saddle point problem. We verify the inf-sup condition

$$\forall q_N \in \mathbb{P}_N(\Omega) \cap M, \qquad \sup_{\mathbf{v}_N \in X_N} \frac{b_N(\mathbf{v}_N, q_N)}{\|v_N\|_{L^2(\Omega)^2}} \geq \gamma \|q_N\|_{H^1(\Omega)}, \tag{4.6}$$

where γ is a positive constant independent from N, by taking $\mathbf{v}_N = \nabla q_N$. Hence, we derive the following theorem.

Theorem 4.1 *Let* $\mathbf{f}$ *be in* $C^0(\overline{\Omega})^2$. *Then problem (4.3), (4.4) has a unique solution* $(\mathbf{u}_N, p_N)$ *satisfying*

$$\|\mathbf{u}_N\|_{L^2(\Omega)^2} + \|p_N\|_{H^1(\Omega)} \leq C\,\|\mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^2}. \tag{4.7}$$

Next, we establish a theorem which implies the convergence of our discretization method.

Theorem 4.2 *Assume that the solution* $(\mathbf{u}, p)$ *of problem (4.3), (4.4) belongs to* $H^s(\Omega)^2 \times H^{s+1}(\Omega)$, $s \geq 0$, *and the data* $\mathbf{f}$ *belongs to* $H^\sigma(\Omega)^2$, $\sigma > 1$, *where* $H^s(\Omega)$, *for non-integral values of* s, *is defined in (2.1). Then, the following estimate holds*

$$\|\mathbf{u} - \mathbf{u}_N\|_{L^2(\Omega)^2} + \|p - p_N\|_{H^1(\Omega)} \leq c\, \Big(N^{-s}(\|\mathbf{u}\|_{H^s(\Omega)^2} + \|p\|_{H^{s+1}(\Omega)}) + N^{-\sigma}\|\mathbf{f}\|_{H^\sigma(\Omega)^2} \Big). \tag{4.8}$$

Proof. From the abstract error estimate for the approximation of saddle-point problems (see [7, Chap. IV]), we derive the following estimate:

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_N\|_{L^2(\Omega)^2} + \|p - p_N\|_{H^1(\Omega)} &\leq c \Big(\inf_{\mathbf{w}_N \in V_N} \|\mathbf{u} - \mathbf{w}_N\|_{L^2(\Omega)^2} \\ + \inf_{\mathbf{v}_N \in X_N} (\|\mathbf{u} - \mathbf{v}_N\|_{L^2(\Omega)^2} &+ \sup_{\mathbf{z}_N \in X_N} \frac{\int_\Omega \mathbf{v}_N(\mathbf{x}) \,.\, \mathbf{z}_N(\mathbf{x})\, d\mathbf{x} - (\mathbf{v}_N, \mathbf{z}_N)_N}{\|\mathbf{z}_N\|_{L^2(\Omega)^2}}) \\ + \inf_{q_N \in \mathbb{P}_N(\Omega) \cap M} (\|p - q_N\|_{H^1(\Omega)} &+ \sup_{\mathbf{z}_N \in X_N} \frac{b(\mathbf{z}_N, q_N) - b_N(\mathbf{z}_N, q_N)}{\|\mathbf{z}_N\|_{L^2(\Omega)^2}}) \\ &+ \sup_{\mathbf{z}_N \in X_N} \frac{\int_\Omega \mathbf{f}(\mathbf{x}) \,.\, \mathbf{z}_N(\mathbf{x})\, d\mathbf{x} - (\mathbf{f}, \mathbf{z}_N)_N}{\|\mathbf{z}_N\|_{L^2(\Omega)^2}} \Big), \end{aligned} \tag{4.9}$$

where V_N is defined by

$$V_N = \{\mathbf{w}_N \in X_N;\ \forall q_N \in \mathbb{P}_N(\Omega) \cap M,\ b_N(\mathbf{w}_N, q_N) = 0\}.$$

Moreover, we recall (see [11, Chap. II, (1.16)]) that

$$\inf_{\mathbf{w}_N \in V_N} \|\mathbf{u} - \mathbf{w}_N\|_{L^2(\Omega)^2} \leq \frac{c}{\gamma} \inf_{\mathbf{v}_N \in X_N} (\|\mathbf{u} - \mathbf{v}_N\|_{L^2(\Omega)^2}).$$

Hence, we derive that, for all $\mathbf{v}_{N-1}$ and $\mathbf{f}_{N-1}$ in $\mathbb{P}_{N-1}(\Omega)^2$ and all $q_{N-1} \in \mathbb{P}_{N-1}(\Omega) \cap M$,

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_N\|_{L^2(\Omega)^2} + \|p - p_N\|_{H^1(\Omega)} \\ &\leq c(\|\mathbf{u} - \mathbf{v}_{N-1}\|_{L^2(\Omega)^2} + \|p - q_{N-1}\|_{H^1(\Omega)} + \|\mathbf{f} - \mathbf{f}_{N-1}\|_{L^2(\Omega)^2} + \|\mathbf{f} - \mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^2}). \end{aligned} \tag{4.10}$$

Then, we choose $\mathbf{v}_{N-1} = \Pi_{N-1}\mathbf{u}$ (resp. $\mathbf{f}_{N-1} = \Pi_{N-1}\mathbf{f}$), that is to say the orthogonal projection of $\mathbf{u}$ (resp. $\mathbf{f}$) on $\mathbb{P}_{N-1}(\Omega)^2$ in $L^2(\Omega)^2$ and $q_{N-1} = \Pi^{1,\Gamma_2}_{N-1}p$, where $\Pi^{1,\Gamma_2}_{N-1}p$ is the orthogonal projection of p on $\mathbb{P}_{N-1}(\Omega) \cap M$ in $H^1(\Omega)$. It remains to prove the estimate, for any $m \geq 1$,

$$\forall p \in H^m(\Omega) \cap M, \ \|p - \Pi^{1,\Gamma_2}_{N-1}p\|_{H^1(\Omega)} \leq C\, N^{1-m}\|p\|_{H^m(\Omega)}. \tag{4.11}$$

On the one hand, this result is obvious for $m = 1$. On the other hand, for $m \geq 2$, we have (see [7, Chap. III]),

$$\begin{aligned} \|p - \Pi^{1,\Gamma_2}_{N-1}p\|_{H^1(\Omega)} &\leq \inf_{r_{N-1}\in\mathbb{P}_{N-1}(\Omega)\cap M} \|p - r_{N-1}\|_{H^1(\Omega)} \\ &\leq \|p - \mathcal{I}_{N-1}p\|_{H^1(\Omega)} \leq C\, N^{1-m}\|p\|_{H^m(\Omega)}. \end{aligned}$$

Then, an interpolation argument (see [11, TH 1.4, page 6]) gives (4.11). Finally, the result follows from (4.10), (4.11) and the classic estimate for the orthogonal projection on $\mathbb{P}_{N-1}(\Omega)$ in $L^2(\Omega)$. ♢

Remark 4.3 *With the choice $X_N = \mathbb{P}_N(\Omega)^2$, problem (4.3, (4.4) can be interpreted as a collocation scheme. Indeed, by integrating by parts in the discrete bilinear form b_N with respect to one of the two variables for each of the two terms of b_N (this process being allowed by the precision of the quadrature rule), and choosing as test functions the Lagrange polynomials associated with the grid points of Ξ_N, it is easily seen that (4.3), (4.4) is equivalent to the set of equations for $\mathbf{u}_N$ in $\mathbb{P}_N(\Omega)^2$ and p_N in $\mathbb{P}_N(\Omega) \cap M$:*

$$\begin{aligned} \mathbf{u}_N(\mathbf{x}) + \nabla p_N(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad & \forall \mathbf{x} \in \Xi_N, \\ \operatorname{div}\mathbf{u}_N(\mathbf{x}) = 0, \quad & \forall \mathbf{x} \in \Xi_N \cap \Omega, \\ \frac{2}{N(N+1)}\operatorname{div}\mathbf{u}_N(\mathbf{x}) = (\mathbf{u}_N\,.\,\mathbf{n})(\mathbf{x}), \quad & \forall \mathbf{x} \in \Xi_N \cap \Gamma^1. \end{aligned}$$

Second spectral discretization

In order to improve the approximation of the condition $\operatorname{div}\mathbf{u} = 0$, we can try to decrease the dimension of the space X_N. So, we choose

$$X_N = \mathbb{P}_{N-1}(\Omega)^2. \tag{4.12}$$

We note that, in this case the forms $b(.,.)$ and $b_N(.,.)$ are equal on $X_N \times \mathbb{P}_N(\Omega)$. It does not appear spurious modes for the pressure, as we can see in the following lemma.

Lemma 4.4 *Let $\mathcal{Z}_N$ be the space*

$$\mathcal{Z}_N = \{q_N \in \mathbb{P}_N(\Omega) \cap M;\ \forall \mathbf{v}_N \in \mathbb{P}_{N-1}(\Omega)^2,\ b(\mathbf{v}_N, q_N) = 0\}.$$

Then $\mathcal{Z}_N = \{0\}$.

Proof. Let q_N be in $\mathcal{Z}_N$. Since $\dfrac{\partial q_N}{\partial x}$ is a polynomial of degree $\leq N-1$ with respect to x, which is orthogonal to $\mathbb{P}_{N-1}(\Omega)$, we can write:

$$q_N(x,y) = \alpha_N(y) + \beta_N(x)L_N(y),$$

with α_N and β_N in $\mathbb{P}_N(\Omega)$. In the same way with y in place of x, we have:

$$q_N(x,y) = \gamma_N(x) + \delta_N(y)L_N(x),$$

with γ_N and δ_N in $\mathbb{P}_N(\Omega)$. Hence, we derive

$$q_N(x,y) = \lambda + \mu L_N(x)L_N(y),$$

where λ and μ are real numbers. But, since $L_N(1) = 0$, the condition:

$$\forall y \in [-1, 1], \; q_N(1, y) = 0$$

implies $\lambda = \mu = 0$. ◊

We have to study the following discrete problem:
Find $\mathbf{u}_N$ in $\mathbb{P}_{N-1}(\Omega)^2$ and p_N in $\mathbb{P}_N(\Omega) \cap M$ such that

$$\forall \mathbf{v}_N \in \mathbb{P}_{N-1}(\Omega)^2, \qquad (\mathbf{u}_N, \mathbf{v}_N)_N + b(\mathbf{v}_N, p_N) = (\mathbf{f}, \mathbf{v}_N)_N, \tag{4.13}$$
$$\forall q_N \in \mathbb{P}_N(\Omega) \cap M, \qquad b(\mathbf{u}_N, q_N) = 0. \tag{4.14}$$

For the inf-sup condition, the choice $\mathbf{v}_N = \nabla q_N$ is no longer available, since $\mathbf{v}_N$ must be in $\mathbb{P}_{N-1}(\Omega)^2$. In fact, in the next lemma, we find a inf-sup constant depending on N.

Lemma 4.5 *There exists a constant $c > 0$ independent on N such that*

$$\forall q_N \in \mathbb{P}_N(\Omega) \cap H^1(\Omega; \Gamma^2), \qquad \sup_{\mathbf{v}_N \in \mathbb{P}_{N-1}(\Omega)^2} \frac{b(\mathbf{v}_N, q_N)}{\|\mathbf{v}_N\|_{L^2(\Omega)^2}} \geq c N^{-1} \|q_N\|_{H^1(\Omega)}. \tag{4.15}$$

On the one hand, we note that

$$q_N^*(x, y) = \sum_{n=0}^{N} a_n(x) L_n(y),$$

where a_N is a polynomial of degree $\leq N - 1$. Then, we have

$$\frac{\partial q_N^*}{\partial x}(x, y) = \Pi_{N-1}(\frac{\partial q_N^*}{\partial x})(x, y) + a'_N(x) L_N(y),$$

which implies, using (2.7) and the inverse inequality (2.10),

$$\|\frac{\partial q_N^*}{\partial x}\|_{L^2(\Omega)} \leq \|\Pi_{N-1}(\frac{\partial q_N^*}{\partial x})\|_{L^2(\Omega)} + c\, N^{\frac{3}{2}} \|a_N\|_{0,\Lambda}. \tag{4.18}$$

On the other hand, in view of (2.8), we can write

$$\Pi_{N-1}(\frac{\partial q_N^*}{\partial y})((x, y) = \sum_{n=0}^{N} \Pi_{N-1}(a_n L'_n)(x, y) = (2N-1) a_N(x) L_{N-1}(y) + r_N(x, y),$$

where $r_N(x, y)$ is a polynomial of $\mathbb{P}_N(\Omega)$ of degree $< N - 1$ with respect to y. The orthogonality properties imply

$$\|\Pi_{N-1}(\frac{\partial q_N^*}{\partial y})\|_{L^2(\Omega)} \geq 2\sqrt{N - \frac{1}{2}} \|a_N\|_{0,\Lambda}. \tag{4.19}$$

By combining this inequality with (4.18), we obtain

$$\|\frac{\partial q_N^*}{\partial x}\|_{L^2(\Omega)} \leq \|\Pi_{N-1}(\frac{\partial q_N^*}{\partial x})\|_{L^2(\Omega)} + c\, N \|\Pi_{N-1}(\frac{\partial q_N^*}{\partial y})\|_{L^2(\Omega)}.$$

In the same way, we have the analogous inequality for $\dfrac{\partial q_N^*}{\partial y}$. Thus, we obtain

$$\|\nabla q_N^*\|_{L^2(\Omega)^2} \leq c\, N \|\Pi_{N-1}(\nabla q_N^*)\|_{L^2(\Omega)^2}. \tag{4.20}$$

Next, the equality (4.16) yields

$$\|\nabla q_N\|_{L^2(\Omega)} \leq \|\nabla q_N^*\|_{L^2(\Omega)} + |\alpha_N| \|\nabla(L_N(x) L_N(y))\|_{L^2(\Omega)},$$

which implies, owing to (2.7) and (2.9),

$$\|\nabla q_N\|_{L^2(\Omega)} \leq \|\nabla q_N^*\|_{L^2(\Omega)} + 2\sqrt{\frac{N(N+1)}{2N+1}}|\alpha_N|. \tag{4.21}$$

It remains to estimate $|\alpha_N|$. First, we have, owing to (4.16),

$$q_N(x,y) = \sum_{n=0}^{N} a_n(x)L_n(y) + \alpha_N L_N(x)L_N(y).$$

But, $\forall y \in [-1,1]$, $q_N(1,y) = 0$, which implies

$$a_n(1) = 0,\ n = 1,\dots,N-1 \qquad \text{and} \qquad a_N(1) + \alpha_N = 0.$$

If we set $a_N(x) = \sum_{k=1}^{N-1} \alpha_k L_k(x)$, we derive

$$\alpha_N = -\sum_{k=0}^{N-1} \alpha_k,$$

and, therefore, thanks to a discrete Cauchy-Schwarz inequality

$$|\alpha_N| \leq (\sum_{k=0}^{N-1} \frac{\alpha_k^2}{k+\frac{1}{2}})^{\frac{1}{2}}(\sum_{k=0}^{N-1}(k+\frac{1}{2}))^{\frac{1}{2}} \leq \frac{\sqrt{2}}{2} N\|a_N\|_{0,\Lambda}.$$

Then, in view of (4.19), we obtain

$$|\alpha_N| \leq \frac{\sqrt{2}\,N}{4\sqrt{N-\frac{1}{2}}}\|\Pi_{N-1}(\nabla q_N^*)\|_{L^2(\Omega)}. \tag{4.22}$$

Finally, (4.20), (4.21) and (4.22) yield

$$\|\nabla q_N\|_{L^2(\Omega)^2} \leq c\,N\|\Pi_{N-1}(\nabla q_N^*)\|_{L^2(\Omega)}$$

which, in view of (4.17), ends the proof. ♢

The bilinear form $(.,.)_N$ and $b(.,.)$ satisfy Brezzi's conditions with respect to $\mathbb{P}_{N-1}(\Omega)^2$ and $\mathbb{P}_N(\Omega) \cap M$ (the bilinear form $(.,.)_N$ is continuous on $\mathbb{P}_{N-1}(\Omega)^2$ and elliptic on $\mathbb{P}_{N-1}(\Omega)$, the bilinear form $b(.,.)$ is continuous on $\mathbb{P}_{N-1}(\Omega) \times (\mathbb{P}_N(\Omega) \cap M)$ and verifies the " inf-sup condition"), see [7, Theorem 2.3, pages 116,117], whence the theorem.

Theorem 4.6 *Let* $\mathbf{f}$ *be in* $C^0(\overline{\Omega})^2$. *Then problem (4.13), (4.14) has a unique solution* $(\mathbf{u}_N, p_N)$ *satisfying*

$$\|\mathbf{u}_N\|_{L^2(\Omega)^2} + N^{-1}\,\|p_N\|_{H^1(\Omega)} \leq C\,\|\mathcal{I}_N\mathbf{f}\|_{L^2(\Omega)^2}. \tag{4.23}$$

We establish the convergence of this second discretization in the following theorem.

Theorem 4.7 *Assume that the solution* $(\mathbf{u}, p)$ *of problem (4.13), (4.14) belongs to* $H^s(\Omega)^2$ $\times H^{s+1}(\Omega)$, $s \geq 0$, *and the data* $\mathbf{f}$ *belongs to* $H^\sigma(\Omega)^2$, $\sigma > 1$. *Then, the following estimate holds*

$$\|\mathbf{u}-\mathbf{u}_N\|_{L^2(\Omega)^2} + N^{-1}\|p-p_N\|_{H^1(\Omega)} \leq c\left(N^{-s}(\|\mathbf{u}\|_{H^s(\Omega)^2} + \|p\|_{H^{s+1}(\Omega)}) + N^{-\sigma}\|\mathbf{f}\|_{H^\sigma(\Omega)^2}\right). \tag{4.24}$$

Proof. The abstract error estimate, analogous to (4.39) but with much simplification because of the exactness of the quadrature formulas, yields

$$\|\mathbf{u}-\mathbf{u}_N\|_{L^2(\Omega)^2} + N^{-1}\|p-p_N\|_{H^1(\Omega)} \leq c\left(\inf_{\mathbf{w}_N \in V_N}\|\mathbf{u}-\mathbf{w}_N\|_{L^2(\Omega)^2}\right.$$

$$+ \inf_{\mathbf{v}_N \in \mathbb{P}_{N-1}(\Omega)^2} \|\mathbf{u} - \mathbf{v}_N\|_{L^2(\Omega)^2} + \inf_{q_N \in \mathbb{P}_N(\Omega) \cap M} \|p - q_N\|_{H^1(\Omega)} + \|\mathbf{f} - \mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^2} \Big),$$

where V_N is now the space

$$V_N = \{\mathbf{w}_N \in \mathbb{P}_{N-1}(\Omega)^2;\ \forall q_N \in \mathbb{P}_N(\Omega) \cap M,\ b(\mathbf{w}_N, q_N) = 0\}.$$

It remains to estimate the term $\inf_{\mathbf{w}_N \in V_N} \|\mathbf{u} - \mathbf{w}_N\|_{L^2(\Omega)^2}$. Since $\mathbf{u}$ is such that $\operatorname{div} \mathbf{u} = 0$ and $\mathbf{u} \,.\, \mathbf{n}_{|\Gamma^1} = 0$, there exist a unique ψ in $H^1(\Omega)$ (see [11, Chap. I and 4]) such that

$$\mathbf{u} = \mathbf{curl}\, \psi \qquad \text{and} \qquad \psi = 0 \text{ on } \Gamma^1.$$

Moreover, if $\mathbf{u}$ belongs to $H^s(\Omega)^2$, we have $\|\psi\|_{H^{s+1}(\Omega)} \leq c \|\mathbf{u}\|_{H^s(\Omega)^2}$. Let us define the operator $\tilde{\pi}_N^1$ (see [7, Chap. II]) on $H^1(\Lambda)$ by

$$\forall \varphi \in H^1(\Lambda),\ (\tilde{\pi}_N^1 \varphi)(\zeta) = (\pi_N^{1,0} \tilde{\varphi})(\zeta) + \varphi(-1) \frac{1-\zeta}{2} + \varphi(1) \frac{1+\zeta}{2}, \tag{4.25}$$

where the function $\tilde{\varphi}$ stands for

$$\tilde{\varphi}(\zeta) = \varphi(\zeta) - \varphi(-1) \frac{1-\zeta}{2} - \varphi(1) \frac{1+\zeta}{2}.$$

Note that the definition of $\tilde{\pi}_N^1$ is available, because $\tilde{\varphi}$ belongs to $H_0^1(\Lambda)$, and that $\pi_N^{1,0} \tilde{\varphi}$ and φ coincide in -1 and 1. In [8, Section 7], the following estimate is proven, for all $r \geq 1$ and all function φ in $H^r(\Lambda)$:

$$|\varphi - \tilde{\pi}_N^1 \varphi|_{1,\Lambda} + N \|\varphi - \tilde{\pi}_N^1 \varphi\|_{0,\Lambda} \leq c\, N^{1-r} \|\varphi\|_{r,\Lambda}. \tag{4.26}$$

Assuming $s \geq 1$, we set

$$R_{N-1}(\mathbf{u}) = \mathbf{curl}\, (\tilde{\pi}_{N-1}^{1(x)} \circ \tilde{\pi}_{N-1}^{1(y)} \psi).$$

Since $(\tilde{\pi}_{N-1}^{1(x)} \circ \tilde{\pi}_{N-1}^{1(y)} \psi)_{|\Gamma^1} = \psi_{|\Gamma^1} = 0$, we can verify that $R_{N-1}(\mathbf{u})$ belongs to V_N and, in view of (4.26), that

$$|\varphi - \tilde{\pi}_{N-1}^{1(x)} \circ \tilde{\pi}_{N-1}^{1(y)} \psi|_{H^1(\Omega)} \leq c\, N^{-s} \|\psi\|_{H^{s+1}(\Omega)}.$$

Finally, we derive, for $s \geq 1$,

$$\inf_{\mathbf{w}_N \in V_N} \|\mathbf{u} - \mathbf{w}_N\|_{L^2(\Omega)^2} \leq \|\mathbf{u} - R_{N-1}(\mathbf{u})\|_{L^2(\Omega)^2} \leq c\, N^{-s} \|\mathbf{u}\|_{H^s(\Omega)^2}.$$

Since we have $\inf_{\mathbf{w}_N \in V_N} \|\mathbf{u} - \mathbf{w}_N\|_{L^2(\Omega)^2} \leq c \|\mathbf{u}\|_{L^2(\Omega)}$, an interpolation argument gives the result of approximation in V_N for any $s \geq 0$ and the estimate of the theorem follows. $\diamondsuit$

Third spectral discretization

The third discretization comes from the variational formulation (3.13), (3.14). We define the space X_N by

$$X_N = \mathbb{P}_N(\Omega)^2 \cap X = \{\mathbf{v}_N \in \mathbb{P}_N(\Omega)^2;\ \mathbf{v}_N \,.\, \mathbf{n}_{|\Gamma^1} = 0\}.$$

Let M_N be a subspace of $\mathbb{P}_N(\Omega)$ that we shall set later. Then, we consider the following discrete problem:

Find $\mathbf{u}_N$ in X_N and p_N in M_N such that

$$\forall \mathbf{v}_N \in X_N, \qquad (\mathbf{u}_N, \mathbf{v}_N)_N + b_N^*(\mathbf{v}_N, p_N) = (\mathbf{f}, \mathbf{v}_N)_N, \tag{4.27}$$

$$\forall q_N \in M_N, \qquad b_N^*(\mathbf{u}_N, q_N) = 0, \tag{4.28}$$

where the form b_N^* is define by

$$\forall \mathbf{v}_N \in \mathbb{P}_N(\Omega)^2,\ \forall q_N \in \mathbb{P}_N(\Omega),\ b_N^*(\mathbf{v}_N, q_N) = -(\operatorname{div} \mathbf{v}_N, q_N)_N. \tag{4.29}$$

In order to choose M_N, we begin to identify the spurious modes for the pressure. These spurious modes for the pressure are derived by elimination from those of classic Stokes problem (see [7, Chap. IV]). In particular, we can verify: $\forall \mathbf{v}_N \in X_N,\ b_N^*(\mathbf{v}_N, q_N) = 0$, for $q_N(x, y) = L_N(x)$ or $L_N(x)L_N(y)$. We obtain the following lemma.

Lemma 4.8 *Let $\mathcal{Z}_N^*$ be the space*

$$\mathcal{Z}_N^* = \{q_N \in I\!P_N(\Omega);\ \forall \mathbf{v}_N \in X_N,\ b_N^*(\mathbf{v}_N, q_N) = 0\}.$$

Then $\mathcal{Z}_N^*$ is spanned by $(L_N(x), L_N(x)L_N(y))$.

Finally, let M_N stand for the orthogonal complement of $\mathcal{Z}_N^*$ for the scalar product in $L^2(\Omega)$ or for the scalar product $(.,.)_N$, owing to (2.11). The inf-sup condition is given in the next lemma.

Lemma 4.9 *There exists a constant $c > 0$ independent from N such that*

$$\forall q_N \in M_N, \qquad \sup_{\mathbf{v}_N \in X_N} \frac{b_N^*(\mathbf{v}_N, q_N)}{\|v_N\|_{H(\operatorname{div};\Omega)}} \geq c\, \|q_N\|_{L^2(\Omega)}. \tag{4.30}$$

Proof. Any function q_N in M_N has the expansion

$$\begin{aligned} q_N(x, y) = &\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} q_{m,n} L_m(x) L_n(y) \\ &+ \sum_{m=0}^{N-1} q_{m,N} L_m(x)(L_N(y) - L_{N-2}(y)) + \sum_{n=1}^{N-1} q_{N,n}(L_N(x) - L_{N-2}(x))L_n(y). \end{aligned}$$

With the convention $L_{-1} = 0$, we choose $\mathbf{w}_N = (w_N, z_N)$ with

$$\begin{aligned} w_N(x, y) = &- \sum_{m=0}^{N-1} \sum_{n=0}^{m} q_{m,n} \frac{L_{m+1}(x) - L_{m-1}(x)}{2m + 1} L_n(y) \\ &- \sum_{m=0}^{N-1} q_{m,N} \frac{L_{m+1}(x) - L_{m-1}(x)}{2m + 1} (L_N(y) - L_{N-2}(y)) \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} z_N(x, y) = &- \sum_{m=0}^{N-1} \sum_{n=m+1}^{N-1} q_{m,n} L_m(x) \frac{L_{n+1}(y) - L_{n-1}(y)}{2n + 1} \\ &- \sum_{n=1}^{N-1} q_{N,n}(L_N(x) - L_{N-2}(x)) \frac{L_{n+1}(y) - L_{n-1}(y)}{2n + 1}. \end{aligned} \tag{4.32}$$

Then, in view of (2.8), we have

$$\operatorname{div} \mathbf{w}_N = -q_N \qquad \text{and} \qquad \mathbf{w}_N \in X_N, \tag{4.33}$$

since $\mathbf{w}_N \,.\, \mathbf{n}_{|\Gamma^1} = z_{N|\Gamma^1} = 0$. As in [6, Chap. IV], we prove

$$\begin{aligned} \|q_N\|_{L^2(\Omega)}^2 \geq c \Bigg(&\sum_{m=0}^{N-1} \sum_{n=0}^{N-1} q_{m,n}^2 \frac{1}{(m + \frac{1}{2})(n + \frac{1}{2})} \\ &+ \frac{1}{N + \frac{1}{2}} \Big(\sum_{m=0}^{N-1} q_{m,N}^2 \frac{1}{m + \frac{1}{2}} + \sum_{n=1}^{N-1} q_{N,n}^2 \frac{1}{n + \frac{1}{2}} \Big) \Bigg). \end{aligned} \tag{4.34}$$

Hence, we derive, in the same way as in [8, Section 24]

$$\|\frac{\partial w_N}{\partial x}\|_{L^2(\Omega)} + \|\frac{\partial z_N}{\partial y}\|_{L^2(\Omega)} \leq c\,\|q_N\|_{L^2(\Omega)}. \tag{4.35}$$

Next, setting $w_N^*(x,y) = w_N(x,y) - q_{0,0}L_1(x) - q_{0,N}L_1(x)(L_N(y) - L_{N-2}(y))$, we note that $w_N^*(\pm 1, y) = 0$ for $-1 \leq y \leq 1$. Then, the Poincaré-Friedrichs inequality, applied with respect to x or y, yields

$$\|w_N^*\|_{L^2(\Omega)} \leq c\|\frac{\partial w_N^*}{\partial x}\|_{L^2(\Omega)} \quad \text{and} \quad \|z_N\|_{L^2(\Omega)} \leq c\|\frac{\partial z_N}{\partial y}\|_{L^2(\Omega)}.$$

Hence, owing to (4.35) and the estimate

$$\sqrt{q_{0,0}^2 + \frac{q_{0,N}^2}{N+\frac{1}{2}}} \leq c\|q_N\|_{L^2(\Omega)},$$

which is derived from (4.34), we obtain

$$\|w_N\|_{L^2(\Omega)} + \|z_N\|_{L^2(\Omega)} \leq c\,\|q_N\|_{L^2(\Omega)}. \tag{4.36}$$

Finally (4.35) and (4.36) imply

$$\|\mathbf{w}_N\|_{H(\mathrm{div}\,;\Omega)} \leq c\,\|q_N\|_{L^2(\Omega)}$$

and, in view of (4.33), the choice $\mathbf{v}_N = \mathbf{w}_N$ in $b_N^*(\mathbf{v}_N, q_N)$ is available and gives the inf-sup condition (4.30). ♢

From the previous lemma, we derive the following theorem.

Theorem 4.10 *Let* $\mathbf{f}$ *be in* $C^0(\overline{\Omega})^2$. *Then problem (4.27), (4.28) has a unique solution* $(\mathbf{u}_N, p_N)$ *satisfying*

$$\|\mathbf{u}_N\|_{H(\mathrm{div}\,;\Omega)} + \|p_N\|_{L^2(\Omega)} \leq C\,\|\mathcal{I}_N\mathbf{f}\|_{L^2(\Omega)^2}. \tag{4.37}$$

Sketch of the proof. From(4.27), we derive $(\mathbf{u}_N, \mathbf{u}_N)_N = (\mathcal{I}_N\mathbf{f}, \mathbf{u}_N)_N$. Then, Owing to $\mathrm{div}\,\mathbf{u}_N = 0$ and (2.13), we derive $\|\mathbf{u}_N\|_{H(\mathrm{div}\,;\Omega)} \leq 3\|\mathcal{I}_N\mathbf{f}\|_{L^2(\Omega)}$. Next, the inf-sup condition (4.30) and (4.27) imply

$$\|p_N\|_{L^2(\Omega)} \leq \frac{1}{c}\sup_{\mathbf{v}_N\in X_N}\frac{b_N^*(\mathbf{v}_N, p_N)}{\|v_N\|_{H(\mathrm{div}\,;\Omega)}} \leq \frac{1}{c}\sup_{\mathbf{v}_N\in X_N}\frac{(\mathcal{I}_N\mathbf{f}, \mathbf{v}_N)_N - (\mathbf{u}_N, \mathbf{v}_N)_N}{\|v_N\|_{H(\mathrm{div}\,;\Omega)}}.$$

Hence, in view of (2.13), we obtain

$$\|p_N\|_{L^2(\Omega)} \leq \frac{18}{c}\|\mathcal{I}_N\mathbf{f}\|_{L^2(\Omega)},$$

which ends the proof. ♢

Next, in the same way as in Theorem 4.2, we prove an optimal error estimate.

Theorem 4.11 *Assume that the solution* $(\mathbf{u}, p)$ *of problem (4.27), (4.28) belongs to* $H^s(\Omega)^2 \times H^s(\Omega)$, $s \geq 0$, *and the data* $\mathbf{f}$ *belongs to* $H^\sigma(\Omega)^2$, $\sigma > 1$. *Then, the following estimate holds*

$$\|\mathbf{u}-\mathbf{u}_N\|_{L^2(\Omega)^2} + \|p-p_N\|_{L^2(\Omega)} \leq c\left(N^{-s}(\|\mathbf{u}\|_{H^s(\Omega)^2} + \|p\|_{H^s(\Omega)}) + N^{-\sigma}\|\mathbf{f}\|_{H^\sigma(\Omega)^2}\right). \tag{4.38}$$

Sketch of the proof. Again, from the abstract error estimate for the approximation of saddle-point problems (see [7, Chap. IV]), we derive the following estimate:

$$\|\mathbf{u}-\mathbf{u}_N\|_{L^2(\Omega)^2}+\|p-p_N\|_{L^2\Omega)}\leq c\,\Bigg(\inf_{\mathbf{w}_N\in V_N}\|\mathbf{u}-\mathbf{w}_N\|_{L^2(\Omega)^2}$$
$$+\inf_{\mathbf{v}_N\in X_N}(\|\mathbf{u}-\mathbf{v}_N\|_{L^2(\Omega)^2}+\sup_{\mathbf{z}_N\in X_N}\frac{\int_\Omega \mathbf{v}_N(\mathbf{x})\,.\,\mathbf{z}_N(\mathbf{x})\,d\mathbf{x}-(\mathbf{v}_N,\mathbf{z}_N)_N}{\|\mathbf{z}_N\|_{L^2(\Omega)^2}})$$
$$+\inf_{q_N\in M_N}(\|p-q_N\|_{L^2(\Omega)}+\sup_{\mathbf{z}_N\in X_N}\frac{b^*(\mathbf{z}_N,q_N)-b^*_N(\mathbf{z}_N,q_N)}{\|\mathbf{z}_N\|_{L^2(\Omega)^2}})$$
$$+\sup_{\mathbf{z}_N\in X_N}\frac{\int_\Omega \mathbf{f}(\mathbf{x})\,.\,\mathbf{z}_N(\mathbf{x})\,d\mathbf{x}-(\mathbf{f},\mathbf{z}_N)_N}{\|\mathbf{z}_N\|_{L^2(\Omega)^2}}\Bigg), \tag{4.39}$$

where V_N is defined by

$$V_N=\{\mathbf{w}_N\in X_N;\ \forall q_N\in M_N,\ b^*_N(\mathbf{w}_N,q_N)=0\}.$$

Moreover, we still have (see [11, CH. II, (1.16)])

$$\inf_{\mathbf{w}_N\in V_N}\|\mathbf{u}-\mathbf{w}_N\|_{L^2(\Omega)^2}\leq\frac{c}{\gamma}\inf_{\mathbf{v}_N\in X_N}(\|\mathbf{u}-\mathbf{v}_N\|_{L^2(\Omega)^2}).$$

We end the proof in the same way as in Theorem 4.2. ♢

Remark 4.12 *As for the first discretization, problem (4.27, (4.28) can be interpreted as a collocation scheme. In the same way, we prove that (4.27), (4.28) is equivalent to the set of equations for* $\mathbf{u}_N$ *in* X_N *and* p_N *in* M_N*:*

$$\mathbf{u}_N(\mathbf{x})+\nabla p_N(\mathbf{x})=\mathbf{f}(\mathbf{x}),\qquad \forall\mathbf{x}\in\Xi_N\cap\Omega,$$
$$\frac{2}{N(N+1)}((\mathbf{u}_N\,.\,\mathbf{n})(\mathbf{x})+\frac{\partial p_N}{\partial n}(\mathbf{x}))=(\mathbf{f}\,.\,\mathbf{n})(\mathbf{x}),\qquad \forall\mathbf{x}\in\Xi_N\cap\Gamma^2,$$
$$\operatorname{div}\mathbf{u}_N(\mathbf{x})=0,\qquad \forall\mathbf{x}\in\Xi_N.$$

Therefore, the discrete solution $\mathbf{u}_N$ is exactly divergence-free, which is important for some applications.

NUMERICAL RESULTS

The convergence of the methods corresponding to the first and third discretizations were tested in a problem of the type (1.1)-(1.4), with homogeneous boundary conditions. Precisely, we tested the convergence of these methods to the exact solution $\mathbf{u}(x,y)=(\pi x^2\cos(\pi y),-2x\sin(\pi y)$ and $p(x,y)=y\sin(\pi x)$, which means that we studied the convergence of these methods for Problem (1.1)-(1.4), with

$$\mathbf{f}(x,y)=(\pi x^2\cos(\pi y)+\pi y\cos(\pi x),-2x\sin(\pi y)+\sin(\pi x))$$

and homogeneous boundary conditions. In addition, we tested the convergence to 0 of the divergence for both methods.

We shall use the Lagrange polynomials. We denote l_r the Lagrange polynomial associated to the Gauss-Lobatto point ξ_r, $0\leq r\leq N$, the expression of which is

$$l_r(x)=\frac{\prod_{j=0,j\neq r}^{N}(x-\xi_j)}{\prod_{j=0,j\neq r}^{N}(\xi_r-\xi_j)}. \tag{5.1}$$

The derivative l'_r verifies the following equalities

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$$\forall r = 0,\dots,N,\ \forall m = 0,\dots,N,\ r \neq m,\ l'_r(\xi_m) = \frac{\prod_{j=0,j\neq m}^{N} (\xi_m - \xi_j)}{\prod_{j=0,j\neq r}^{N} (\xi_r - \xi_j)} (\frac{1}{\xi_m - \xi_r}), \tag{5.2}$$

$$\forall r = 0,\dots,N,\ l'_r(\xi_r) = \sum_{j=0,j\neq r}^{N} \frac{1}{\xi_r - \xi_j}. \tag{5.3}$$

Uzawa's algorithm

Problem (4.3), (4.4), Problem (4.13), (4.14) and Problem (4.27), (4.28) are equivalent to a linear system of the type :

$$\begin{cases} MU + DP = MF, \\ D^T U = 0 \end{cases}. \tag{5.4}$$

The unkowns are the vectors U and P which represent respectively the velocity and the pressure values on a given grid points. The data f is representated by the vector F on the same grid points. The diagonal matrix M is the weight matrix, while the matrix D is associated to the form b_N for the first and second spectral discretizations and to the form b^*_N for the third spectral discretization and D^T is the transposed matrix of D.

Uzawa's algorithm consists in rewriting the first equation of system (5.4) as: $U = F - M^{-1}DP$ and substituting in the second equation. We obtain a new equation for the pressure P:

$$(D^T M^{-1} D)P = D^T F. \tag{5.5}$$

Next, we solve this symetric system either directly if the matrix $D^T M^{-1} D$ is invertible or by diagonalizing the matrix $D^T M^{-1} D$ if not, because the spurious modes correspond to eigenvalues of the matrix equal to zero. Next, we compute the velocity via the formula

$$U = F - M^{-1}DP$$

and its divergence by multiplying U on the left by the matrix D^T. Finally, we test the convergence to 0 of its divergence.

Implementation of the rst discretization

We take $(l_j(x)l_k(y), 0)$, $(0, l_j(x)l_k(y))$, $0 \leq i,j \leq N$ as a basis of X_N and $l_r(x)l_s(y)$, $1 \leq r \leq N-1$, $0 \leq s \leq N$ as a basis of $P_N(\Omega) \cap M$. The unknowns are the velocity $\mathbf{u}_N = (u^1_N, u^2_N)$ and the pressure p_N. For $j = 1, 2$, for $0 \leq r, s \leq N$, we denote $U^{j,N}_{r,s} = u^j_N(\xi_r, \xi_s)$, $F^{j,N}_{r,s} = f_j(\xi_r, \xi_s)$, where $\mathbf{f} = (f_1, f_2)$ represents the data, and, for $1 \leq r \leq N-1$, for $0 \leq s \leq N$, we denote $P^N_{r,s} = p_N(\xi_r, \xi_s)$. So, we have

$$u^j_N(x,y) = \sum_{r,s=0}^{N} U^{j,N}_{r,s} l_r(x) l_s(y),\ j = 1,2 \text{ and } p_N(x,y) = \sum_{r=1}^{N-1}\sum_{s=0}^{N} P^N_{r,s} l_r(x) l_s(y). \tag{5.6}$$

Let us define the $(N+1)^2 \times (N+1)^2$ diagonal matrix $\tilde{M} = (m_{(j,k),(r,s)})_{0\leq j,k,r,s\leq N}$ with

$$m_{(j,k),(r,s)} = \begin{cases} 0 & \text{if } (r,s) \neq (j,k) \\ \rho_j \rho_k & \text{if } (r,s) = (j,k), \end{cases} \tag{5.7}$$

and the $2(N+1)^2 \times 2(N+1)^2$ diagonal matrix

$$M = \begin{pmatrix} \tilde{M} & 0 \\ 0 & \tilde{M} \end{pmatrix}.$$

By setting $\mathbf{v}_N(x,y) = (l_j(x)l_k(y), 0)$, for $0 \leq j,k \leq N$ and, next, $\mathbf{v}_N(x,y) = (0, l_j(x)l_k(y)$ in (4.3), (4.4), we obtain the matrix system (5.4) where the column matrices U, P and F are defined by

$$U = \begin{pmatrix} U^{1,N}_{j,k} \\ U^{2,N}_{j,k} \end{pmatrix},\ P = (P^N_{r,s}) \text{ and } F = \begin{pmatrix} F^{1,N}_{j,k} \\ F^{2,N}_{j,k} \end{pmatrix},\ 0 \le j,k,s \le N,\ 1 \le r \le N-1,$$

and, with $(.,.)_N$ defined by (4.1), the $2(N+1)^2 \times (N^2-1)$ matrix D by

$$D = \begin{pmatrix} D^{1,N} \\ D^{2,N} \end{pmatrix} \text{ with } D^{1,N} = ((l_j(x)l_k(y), l'_r(x)l_s(y))_N),\ D^{2,N} = ((l_j(x)l_k(y), l_r(x)l'_s(y))_N),$$

where (j,k) represents the row index, (r,s) the column index, $0 \le i,j,s \le N$, $1 \le r \le N-1$, for the matrices $D^{1,N}$ and $D^{2,N}$. Note that U and F are $2(N+1)^2 \times 1$ matrices and P is a $(N^2-1) \times 1$ matrix.

Proposition 5.1 *The $(N^2-1) \times (N^2-1)$ square matrix $D^T M^{-1} D$ is invertible.*

Proof. We just have to prove that the rank of the matrix D is N^2-1. Let us assume that the rank of the matrix D is strictly smaller than N^2-1. Then, there exist a sequence of real number $(q_{r,s})$, $1 \le r \le N-1$, $0 \le s \le N$, where all the real numbers $q_{r,s}$ are not equal to zero, such that

$$\sum_{r=1}^{N-1} \sum_{s=0}^{N} q_{r,s} D_{r,s} = 0,$$

where $D_{r,s}$ are the column vectors of the matrix D. Setting $q = \sum_{r=1}^{N-1} \sum_{s=0}^{N} q_{r,s} l_r(x) l_s(y)$, the previous equality is equivalent to

$$\forall \mathbf{v}_N \in X_N,\ b(\mathbf{v}_N, q) = 0,$$

which is in contradiction with the property that there is no spurious mode. ♢

We have to compute the matrix $D^T M^{-1} D = (b_{(t,u),(r,s)})$, $1 \le r,t \le N-1$, $0 \le s,u \le N$. Owing to the previous expression of the matrices D and M, we obtain

$$\begin{aligned} b_{(t,u),(r,s)} = & \sum_{m,k=0}^{N} \frac{1}{\rho_m \rho_k} (l_m(x)l_k(y), l'_t(x)l_u(y))_N (l_m(x)l_k(y), l'_r(x)l_s(y))_N \\ & + \sum_{m,k=0}^{N} \frac{1}{\rho_m \rho_k} (l_m(x)l_k(y), l_t(x)l'_u(y))_N (l_m(x)l_k(y), l_r(x)l'_s(y))_N. \end{aligned}$$

Next, we change the numbering for the matrix $D^T M^{-1} D$. Let us define the mapping $\boldsymbol{\varphi}$ by

$$\forall (r,s),\ 1 \le r \le N-1,\ 0 \le s \le N,\ \boldsymbol{\varphi}(r,s) = 1 + (r-1)(N+1) + s. \tag{5.8}$$

Note that $\boldsymbol{\varphi}$ is a one to one mapping from $\{1,\dots,N-1\} \times \{0,\dots,N\}$ to $\{1,\dots,N^2-1\}$ and we note

$$\forall 1 \le i \le N^2-1,\ \boldsymbol{\varphi}^{-1}(i) = (\psi_1(i), \psi_2(i)). \tag{5.9}$$

Note that $\psi_1(i)-1$ and $\psi_2(i)$ are respectively the quotient and the remainder of the euclidian division of $i-1$ by $N+1$. Then, we can denote

$$D^T M^{-1} D = (a_{i,j})_{1 \le i,j \le N^2-1} \text{ with } a_{i,j} = b_{(t,u),(r,s)}, \tag{5.10}$$

where $(t,u) = (\psi_1(i), \psi_2(i))$ and $(r,s) = (\psi_1(j), \psi_2(j))$.

Computing the elements $a_{i,j}$ of the matrix $D^T M^{-1} D$ yields

1) if $\psi_1(i) \ne \psi_1(j)$ and $\psi_2(i) \ne \psi_2(j)$

$a_{i,j} = 0$

2) if $\psi_1(i) \ne \psi_1(j)$ and $\psi_2(i) = \psi_2(j)$

$a_{i,j} = \rho_{\psi_2(i)} \sum_{m=0}^{N} \rho_m l'_{\psi_1(i)}(\xi_m) l'_{\psi_1(j)}(\xi_m)$

3) if $\psi_1(i) = \psi_1(j)$ and $\psi_2(i) \ne \psi_2(j)$

$a_{i,j} = \rho_{\psi_1(i)} \sum_{m=0}^{N} \rho_m l'_{\psi_2(i)}(\xi_m) l'_{\psi_2(j)}(\xi_m)$

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4) if $\psi_1(i) = \psi_1(j)$ and $\psi_2(i) = \psi_2(j)$, that is to say i=j
$a_{i,i} = \rho_{\psi_2(i)} \sum_{m=0}^{N} \rho_m (l'_{\psi_1(i)})^2 + \rho_{\psi_1(i)} \sum_{m=0}^{N} \rho_m (l'_{\psi_2(i)})^2.$
Thus, most of elements of the matrix $D^T M^{-1} D$ are equal to zero.

Next, we determine the column matrix $D^T F = (c_{i,1})$ by

$$\forall 1 \le i \le N^2 - 1,\ c_{i,1} = \rho_{\psi_2(i)} \sum_{m=0}^{N} \rho_m l'_{\psi_1(i)}(\xi_m) F^{1,N}_{m,\psi_2(i)} + \rho_{\psi_1(i)} \sum_{m=0}^{N} \rho_m l'_{\psi_2(i)}(\xi_m) F^{2,N}_{\psi_1(i),m}.$$

Hence, owing to (5.2) and (5.3), we compute the list $[l'_k(\xi_m),\ 0 \le k, m \le N]$, which allows us to determine the elements of the matrices $D^T M^{-1} D$ and $D^T F$. Since the matrix $D^T M^{-1} D$ is invertible, the equation $(D^T M^{-1} D) X = D^T F$ has a unique solution X. Then we derive easily the column matrix $P = (P^N_{r,s})$ such that $P^N_{r,s} = X_{\boldsymbol{\varphi}(r,s)}$, $1 \le r \le N-1$, $0 \le s \le N$ and, next, the column matrix U, thanks to the relation $U = F - M^{-1} D P$. Setting $P^N_{0,t} = P^N_{N,t} = 0$ for $0 \le t \le N$ in accordance with the boundary conditions, we obtain

$$U^{1,N}_{m,k} = F^{1,N}_{m,k} - \sum_{u=1}^{N-1} l'_u(\xi_m) P^N_{u,k},\ U^{2,N}_{m,k} = F^{2,N}_{m,k} - \sum_{t=0}^{N} l'_t(\xi_k) P^N_{m,t}.$$

Finally, we derive

$$(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N = \sum_{i,j=0}^{N} \rho_i \rho_j \Big(\sum_{m=0}^{N} (U^{1,N}_{m,j} l'_m(\xi_i) + U^{2,N}_{i,m} l'_m(\xi_j)) \Big)^2,$$

$$(\mathbf{u} - \mathbf{u}_N, \mathbf{u} - \mathbf{u}_N)_N = \sum_{i,j=0}^{N} \rho_i \rho_j (u_1(\xi_i, \xi_j) - U^{1,N}_{i,j})^2 + (u_2(\xi_i, \xi_j) - U^{2,N}_{i,j})^2),$$

$$(p - p_N, p - p_N)_N = \sum_{i=1}^{N-1} \sum_{j=0}^{N} \rho_i \rho_j (p(\xi_i, \xi_j) - P^N_{i,j})^2.$$

We give the values of $(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac12}$, $(p - p_N, p - p_N)_N^{\frac12}$ and $(\mathbf{u} - \mathbf{u}_N, \mathbf{u} - \mathbf{u}_N)_N^{\frac12}$ for N between 4 and 21.

N	4	5	6	7	8	9
$(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac12}$	6,85	0,27	1,12	0,0155	0,080	$5,16.10^{-4}$
$(p-p_N, p-p_N)_N^{\frac12}$	0,06	0,02	$2,7.10^{-3}$	$7,75.10^{-4}$	$7,59.10^{-5}$	$1,74.10^{-5}$
$(\mathbf{u}-\mathbf{u}_N, \mathbf{u}-\mathbf{u}_N)_N^{\frac12}$	1,07	0,045	0,010	$1,56.10^{-3}$	$4,65.10^{-3}$	$3,52.10^{-5}$

N	10	11	12	13	14	15
$(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac12}$	$3,14.10^{-3}$	$1,11.10^{-5}$	$7,86.10^{-5}$	$1,70.10^{-7}$	$1,36.10^{-6}$	$1,92.10^{-9}$
$(p-p_N, p-p_N)_N^{\frac12}$	$1,44.10^{-6}$	$2,78.10^{-7}$	$1,96.10^{-8}$	$3,27.10^{-9}$	$2,03.10^{-10}$	$2,97.10^{-11}$
$(\mathbf{u}-\mathbf{u}_N, \mathbf{u}-\mathbf{u}_N)_N^{\frac12}$	$1,33.10^{-4}$	$5,6.10^{-7}$	$2,56.10^{-6}$	$6,62.10^{-9}$	$3,54.10^{-8}$	$6,03.10^{-11}$

N	16	17	18	19	20	21
$(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac12}$	$1,73.10^{-8}$	$1,68.10^{-11}$	$1,69.10^{-10}$	$1,02.10^{-12}$	$3,79.10^{-12}$	$1,91.10^{-11}$
$(p-p_N, p-p_N)_N^{\frac12}$	$1,64.10^{-12}$	$2,15.10^{-13}$	$1,06.10^{-14}$	$5,50.10^{-15}$	$1,09.10^{-14}$	$1,06.10^{-14}$
$(\mathbf{u}-\mathbf{u}_N, \mathbf{u}-\mathbf{u}_N)_N^{\frac12}$	$3,70.10^{-10}$	$4,37.10^{-13}$	$3,03.10^{-12}$	$3,03.10^{-14}$	$9,03.10^{-14}$	$2,69.10^{-13}$

Table 4.1

We shall comment these results on comparing them with these ones of the third discretization.

Implementation of the first discretization

We only sketch the method because the first and third discretizations are rather similar, but we shall point out the differences between both discretizations.

First, the space X_N is not the same, because boundary conditions are taken in account. Thus, we take $(l_j(x)l_k(y), 0)$, $(0, l_s(x)l_r(y))$, $0 \le i, j, s \le N$, $1 \le r \le N-1$ as a basis of X_N.

Second, we shall compute the pressure p_N as the orthogonal projection on M_N of an element of $\mathbb{P}_N(\Omega)$ and we take $l_r(x)l_s(y)$, $0 \le r, s \le N$ as a basis of $\mathbb{P}_N(\Omega)$. For $0 \le r, s \le N$, we denote $U^{1,N}_{r,s} = u^1_N(\xi_r, \xi_s)$, $P^N_{r,s} = p_N(\xi_r, \xi_s)$ and for $0 \le r \le N$, $1 \le s \le N-1$, we denote $U^{2,N}_{r,s} = u^2_N(\xi_r, \xi_s)$. Let us define the matrix $\tilde{M}_1 = \tilde{M}$, where $\tilde{M}$ is given by (5.7), the $(N^2-1) \times (N^2-1)$ matrix $\tilde{M}_2 = (m_{(j,k),(r,s)})$ with

$$m_{(j,k),(r,s)} = \begin{cases} 0 & \text{if } (r,s) \ne (j,k) \\ \rho_j \rho_k & \text{if } (r,s) = (j,k), \end{cases} \quad 0 \le j, r \le N,\ 1 \le k, s \le N-1$$

and the $2N(N+1) \times 2N(N+1)$ diagonal matrix

$$M_* = \begin{pmatrix} \tilde{M}_1 & 0 \\ 0 & \tilde{M}_2 \end{pmatrix}.$$

In the same way as the first discretization, we derive the following matrix system

$$\begin{cases} M_* U + D_* P = M_* F, \\ D_*^T U = 0 \end{cases}, \tag{5.11}$$

where the column matrices U, P and F are defined by

$$U = \begin{pmatrix} U^{1,N}_{j,k} \\ U^{2,N}_{t,u} \end{pmatrix},\ P = (P^N_{r,s}) \text{ and } F = \begin{pmatrix} F^{1,N}_{j,k} \\ F^{2,N}_{t,u} \end{pmatrix},\ 0 \le j, k, r, s, t \le N,\ 1 \le u \le N-1$$

and the $2N(N+1) \times (N+1)^2$ matrix D_* by

$$D_* = \begin{pmatrix} D^{1,N}_* \\ D^{2,N}_* \end{pmatrix} \text{ with } D^{1,N}_* = (-(l'_j(x)l_k(y), l_r(x)l_s(y))_N),\ D^{2,N}_* = (-(l_t(x)l'_u(y), l_r(x)l_s(y))_N).$$

We have to compute the matrix $D_*^T M_*^{-1} D_* = (b^*_{(t,u),(r,s)})$, $0 \le r, s, t, u \le N$. In the same way as the first discretization, we obtain

$$\begin{aligned} b^*_{(t,u),(r,s)} = & \sum_{m,k=0}^{N} \frac{1}{\rho_m \rho_k} (l'_m(x)l_k(y), l_t(x)l_u(y))_N (l'_m(x)l_k(y), l_r(x)l_s(y))_N \\ & + \sum_{m=0}^{N} \sum_{k=1}^{N-1} \frac{1}{\rho_m \rho_k} (l_m(x)l'_k(y), l_t(x)l_u(y))_N (l_m(x)l'_k(y), l_r(x)l_s(y))_N. \end{aligned}$$

Next, as in the first discretization, we change the numbering for the matrix $D_*^T M_*^{-1} D_*$. Let us define the mapping $\boldsymbol{\varphi}_*$ by

$$\forall (r,s),\ 0 \le r, s \le N,\ \boldsymbol{\varphi}_*(r,s) = r(N+1) + s + 1. \tag{5.12}$$

Note that $\boldsymbol{\varphi}_*$ is a one to one mapping from $\{0, \dots, N\}^2$ to $\{1, \dots, (N+1)^2\}$ and we note

$$\forall 1 \le i \le N^2 - 1,\ \boldsymbol{\varphi}_*^{-1}(i) = (\psi^*_1(i), \psi^*_2(i)). \tag{5.13}$$

Note that $\psi^*_1(i)$ and $\psi^*_2(i)$ are respectively the quotient and the remainder of the euclidian division of $i-1$ by $N+1$. Then, we can denote

$$D_*^T M_*^{-1} D_* = (a^*_{i,j})_{1 \le i,j \le (N+1)^2} \text{ with } a^*_{i,j} = b^*_{(t,u),(r,s)}, \tag{5.14}$$

where $(t,u) = (\psi_1^*(i), \psi_2^*(i))$ and $(r,s) = (\psi_1^*(j), \psi_2^*(j))$.

Computing the elements $a_{i,j}^*$ of the matrix $D_*^T M_*^{-1} D_*$ yields

1) if $\psi_1^*(i) \neq \psi_1^*(j)$ and $\psi_2^*(i) \neq \psi_2^*(j)$
$a_{i,j}^* = 0$
2) if $\psi_1^*(i) \neq \psi_1^*(j)$ and $\psi_2^*(i) = \psi_2^*(j)$
$a_{i,j}^* = \rho_{\psi_1^*(i)} \rho_{\psi_1^*(j)} \rho_{\psi_2^*(i)} \sum_{m=0}^{N} \frac{1}{\rho_m} l'_m(\xi_{\psi_1^*(i)}) l'_m(\xi_{\psi_1^*(j)})$
3) if $\psi_1^*(i) = \psi_1^*(j)$ and $\psi_2^*(i) \neq \psi_2^*(j)$
$a_{i,j}^* = \rho_{\psi_2^*(i)} \rho_{\psi_2^*(j)} \rho_{\psi_1^*(i)} \sum_{m=1}^{N-1} \frac{1}{\rho_m} l'_m(\xi_{\psi_2^*(i)}) l'_m(\xi_{\psi_2^*(j)})$
4) if $\psi_1^*(i) = \psi_1^*(j)$ and $\psi_2^*(i) = \psi_2^*(j)$, that is to say i=j
$a_{i,i}^* = (\rho_{\psi_1^*(i)})^2 \rho_{\psi_2^*(i)} \sum_{m=0}^{N} \frac{1}{\rho_m} (l'_m(\xi_{\psi_1^*(i)}))^2 + (\rho_{\psi_2^*(i)})^2 \rho_{\psi_1^*(i)} \sum_{m=1}^{N-1} \frac{1}{\rho_m} (l'_m(\xi_{\psi_2^*(i)}))^2.$
zero.

Next, we determine the column matrix $D_*^T F = (c_{i,1}^*)$ by

$$\forall 1 \leq i \leq (N+1)^2,\ c_{i,1} = -\rho_{\psi_1(i)} \rho_{\psi_2(i)} \Big(\sum_{m=0}^{N} l'_m(\xi_{\psi_1^*(i)}) F_{m,\psi_2(i)}^{1,N} + \sum_{m=1}^{N-1} l'_k(\xi_{\psi_2^*(i)}) F_{\psi_1(i),m}^{2,N} \Big).$$

Now, we deal with the main difference between both discretizations. In the third discretization, the matrix $D_*^T M_*^{-1} D_*$ is not invertible and the computation of the pressure is more complicated. First, let $q_N = \sum_{t,u=0}^{N} q_{t,u} l_t(x) l_u(y)$ be a spurious mode, that is to say an element of the space $\mathcal{Z}_N^*$ defined in Lemma 4.8. In the same way as in the proof of Proposition 5.1, considering that the matrices $D_*^T M_*^{-1} D_*$ and D_* have the same rank, we obtain the following equivalences

$$q_N \in \mathcal{Z}_N^* \iff D_* Q = 0 \iff (D_*^T M_*^{-1} D_*) Q = 0,$$

where Q is the column vector $(q_{t,u})_{0 \leq t,u \leq N}$. We can consider the matrix $D_*^T M_*^{-1} D_*$ as the matrix of a linear mapping f from the vector space $\mathbb{P}_N(\Omega)$ into itself equipped with the basis $\mathcal{B} = (l_i(x) l_j(y))_{0 \leq i,j \leq N}$. Therefore, $\mathcal{Z}_N^*$ is the eigenspace associated to the eigenvalue equal to 0 of the linear mapping f or, equivalently, of its matrix $D_*^T M_*^{-1} D_*$. Note that this basis is orthonormal for the scalar product $(.,.)_N$. We can diagonalize the positive symmetric matrix $D_*^T M_*^{-1} D_*$ and, thus, there exist a diagonal matrix Λ and an orthogonal matrix R such that $D_*^T M_*^{-1} D_* = R \Lambda R^{-1}$ and such that the diagonal elements of Λ, that is to say $(\lambda_{i,i})_{1 \leq i \leq (N+1)^2}$ are in increasing order. Note that the matrix Λ is the matrix of f in a basis $\mathcal{B}' = (h_i)_{1 \leq i \leq (N+1)^2}$ of $\mathbb{P}_N(\Omega)$, which is orthonormal for the scalar product $(.,.)_N$, the elements of which are the eigenvectors of the matrix $D_*^T M_*^{-1} D_*$. Let P' be the column matrix the elements of which are the components of the pressure p_N in the new basis $\mathcal{B}'$ of $\mathbb{P}_N(\Omega)$. We have $P' = R^T P = R^{-1} P$ and the following equivalence

$$(D_*^T M_*^{-1} D_)P = D_*^T F \iff \Lambda P' = R^T D_*^T F. \tag{5.15}$$

Since $\mathcal{Z}_N^*$ is a two dimensions space, $\mathcal{B}'_0 = (h_1, h_2)$ is the basis of $\mathcal{Z}_N^*$ and $(h_i)_{3 \leq i \leq (N+1)^2}$ is a basis of $M_N = (\mathcal{Z}_N^*)^\perp$. Therefore, we determine the column vectors P' and P by

$$P'_1 = P'_2 = 0,\ P'_i = \frac{1}{\lambda_{i,i}} (R^T D_*^T F)_i,\ 3 \leq i \leq (N+1)^2 \text{ and } P = RP', \tag{5.16}$$

and we have $P_{r,s}^N = P_{\boldsymbol{\varphi}_*(r,s)}$, $0 \leq r,s \leq N$. In the same way as for the first discretization, we derive the column matrix U by the relation $U = F - M_*^{-1} D_* P$, which gives, for $0 \leq j,k,t \leq N$, $1 \leq u \leq N-1$,

$$U_{j,k}^{1,N} = F_{j,k}^{1,N} + \frac{1}{\rho_j} \sum_{r=0}^{N} \rho_r l'_j(\xi_r) P_{r,k}^N,\ U_{t,u}^{2,N} = F_{t,u}^{2,N} + \frac{1}{\rho_u} \sum_{r=0}^{N} \rho_r l'_u(\xi_r) P_{t,r}^N.$$

Finally, considering the boundary conditions, we set $U_{t,0}^{2,N} = U_{t,N}^{2,N} = 0$, $0 \leq t \leq N$, and we obtain the same formulas as in the first discretization for $(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N$, $(\mathbf{u} - \mathbf{u}_N, \mathbf{u} - \mathbf{u}_N)_N$ and $(p - p_N, p - p_N)_N$. We also give the values of $(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac{1}{2}}$, $(p - p_N, p - p_N)_N^{\frac{1}{2}}$ and $(\mathbf{u} - \mathbf{u}_N, \mathbf{u} - \mathbf{u}_N)_N^{\frac{1}{2}}$ for N between 4 and 21.

N	4	5	6	7	8	9
$(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac{1}{2}}$	$1,84.10^{-16}$	$5,45.10^{-15}$	$3,60.10^{-16}$	$3,05.10^{-15}$	$8,76.10^{-16}$	$1,94.10^{-14}$
$(p-p_N, p-p_N)_N^{\frac{1}{2}}$	0,246	0,016	$0,019$	$0,0039$	$8,43.10^{-4}$	$1,55.10^{-3}$
$(\mathbf{u}-\mathbf{u}_N, \mathbf{u}-\mathbf{u}_N)_N^{\frac{1}{2}}$	0,62	0,043	0,054	$1,54.10^{-3}$	$2,48.10^{-3}$	$3,54.10^{-5}$

N	10	11	12	13	14	15
$(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac{1}{2}}$	$1,24.10^{-14}$	$9,28.10^{-15}$	$2,7.10^{-13}$	$9,68.10^{-13}$	$4,25.10^{-13}$	$3,43.10^{-13}$
$(p-p_N, p-p_N)_N^{\frac{1}{2}}$	$2,3.10^{-5}$	$8,25.10^{-4}$	$4,33.10^{-7}$	$4,8.10^{-4}$	$5,9.10^{-9}$	$3,1.10^{-4}$
$(\mathbf{u}-\mathbf{u}_N, \mathbf{u}-\mathbf{u}_N)_N^{\frac{1}{2}}$	$7,0.10^{-5}$	$5,7.10^{-7}$	$1,34.10^{-6}$	$6,79.10^{-9}$	$1,85.10^{-8}$	$6,23.10^{-11}$

N	16	17	18	19	20	21
$(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac{1}{2}}$	$4,38.10^{-13}$	$2,24.10^{-12}$	$3,59.10^{-13}$	$1,9.10^{-12}$	$8,71.10^{-13}$	$2,68.10^{-12}$
$(p-p_N, p-p_N)_N^{\frac{1}{2}}$	$6,11.10^{-11}$	$2,05.10^{-4}$	$4,96.10^{-13}$	$1,44.10^{-4}$	$3,24.10^{-14}$	$1,04.10^{-4}$
$(\mathbf{u}-\mathbf{u}_N, \mathbf{u}-\mathbf{u}_N)_N^{\frac{1}{2}}$	$1,93.10^{-10}$	$4,53.10^{-13}$	$1,57.10^{-12}$	$3,66.10^{-14}$	$1,12.10^{-13}$	$2,38.10^{-13}$

Table 4.2

Logarithm to the basis 10 of $(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac{1}{2}}$
in function of N for the first and third discretizations

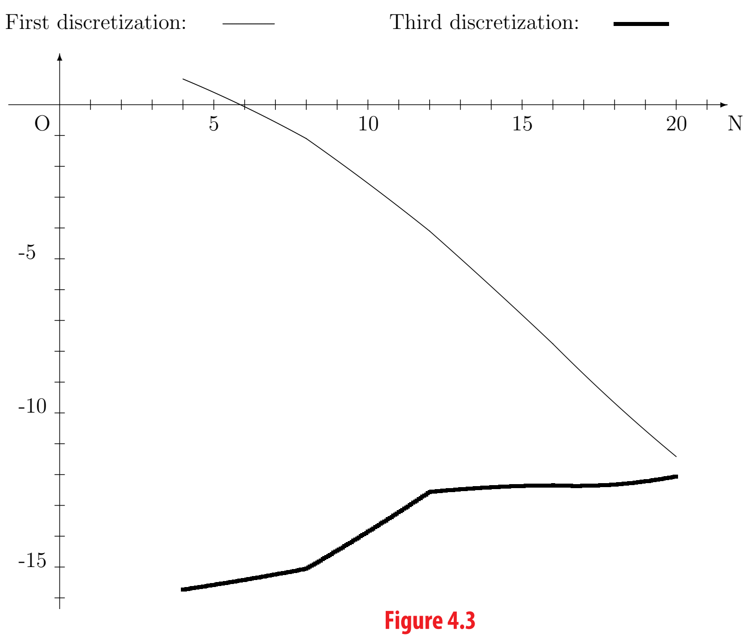

Figure 4.3

CONCLUSION

Continuous problem and discretizations

We studied the Darcy problem by defining two equivalent variational formulations corresponding to two different bilinear forms b and b^*. The first one requests a more regular pressure, which is bounded in $H^1(\Omega)$. The second one is less classic and was introduced by A. Quarteroni and A. Valli (see [14]). This second variational formulation is important because it allowed us to constuct a spectral method which gives a divergence-free discrete solution.

Moreover, by studying a mixed problem of Dirichlet-Neumann for the Laplace operator, we proved regularity results with the solution$(\mathbf{u}, p)$ in $H^1(\Omega)^2 \times H^2(\Omega)$ as long as the data is regular enough.

The first variational formulation led us to two spectral discretizations. In the first one, the discrete solution is not divergence-free, which is a disadvantage for some applications. However, we obtain a fully optimal error estimate for the velocity and the pressure. In the second discretization, it does not appear spurious modes for the pressure. Moreover, because of the exactness of the quadrature formulas, the error estimate is easier to obtain. However, the error estimate for the pressure is not optimal, because the inf-sup constant depends on N.

The second variational formulation led to a third spectral discretization. There are spurious modes, which complicate the study, but the discrete velocity is divergence-free, which is important when the system is a stage of solving of a time-dependent problem, and the error estimate for the velocity and the pressure is fully optimal. In conclusion, this third discretization is the best discretization and we will only use it hereafter.

Comparison of both spectral discretizations

Tables 4.1 and 4.2 test the convergence of respectively the first and third discretizations to the exact solutions. For the first discretization, we see the convergence to zero of $(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac{1}{2}}$, about 10^{-1} for $N = 8$ and about 10^{-12} for $N = 19$. Concerning the third discretization, we see that the quantity $(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac{1}{2}}$ is very small for all values of N, about 10^{-16} for $N = 4$ and about 10^{-12} for $N = 20$. Thus, we verify that, in the third discretization, the discrete solution $\mathbf{u}_N$ is exactly divergence-free, which is important, as we saw previously. This property appears clearly in Figure 4.3, which represents the logarithm to the basis 10 of the quantity $(\operatorname{div} \mathbf{u}_N, \operatorname{div} \mathbf{u}_N)_N^{\frac{1}{2}}$ as a function of N, for even N from 4 to 20, for both discretizations.

Regarding the velocity, the discrete solution $\mathbf{u}_N$ converges fast to the exact solution $\mathbf{u}$ for both discretizations. For the pressure, we also obtain fast convergence, except for odd N in the third discretization.

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