AVANCES
EN CIENCIAS E INGENIERÍAS
ARTÍCULO/ARTICLE SECCIÓN/SECTION A

On a class of embedded cubature formulae on the simplex
Sobre una clase de fórmulas de cubicación encajadas en el símplex

Francesco A. Costabile[1] **and Luca Guzzardi**[2*]

[1] *Dipartimento di Matematica, Uni Cal, Rende, Cosenza, Italy.*
[1] *Universidad San Francisco de Quito, Mathematics Department. Colegio de Ciencias e Ingenierías, El Politécnico Cumbayá, Diego de Robles y Vía Interoceánica. Ecuador*
[*] *Autor principal/Corresponding author, e-mail: lguzzardi@gmail.com*

Editado por/Edited by: Cesar Zambrano, Ph.D.
Recibido/Received: 25/04/2014. Aceptado/Accepted: 28/04/2014.
Publicado en línea/Published on Web: 13/06/2014. Impreso/Printed: 13/06/2014.

Abstract

In this paper we investigate a class of embedded cubature formulae on the simplex announced in [1]. Here we recall the class of formulae, we introduce the remainder and we give an estimation of this, we also investigate the convergence. Some numerical examples are given.
Mathematical subject classification: Primary: 65D32

Keywords. approximation by rational functions, cubature, simplex.

Resumen

En este trabajo se investiga una clase de fórmulas de cubicación encajados en el símplex anunciado en [1]. Aquí recordamos la clase de fórmulas, se introduce el resto y damos un estimado de lo mencionado y también investigamos la convergencia. Se dan algunos ejemplos numéricos.

Palabras Clave. aproximación por funciones racionales, cubicación, símplex

1 Introduction

Let T be the standard triangle in $\mathbb{R}^2$, i.e.

$$T = \{(x, y) : x > 0 \quad x + y \leq 1\} \tag{1}$$

and let $f : T \to \mathbb{R}^2$ be a sufficiently smooth function. The numerical calculation of the integral

$$I_T[f] := \int_T f \, \mathrm{d}x \, \mathrm{d}y \tag{2}$$

by means of cubature formula is of special interest. Widely used cubature formulae for (2) are of type

$$Q_n[f] := \sum_{i=1}^{n} w_i f(x_i, y_i) \qquad w_i \in \mathbb{R} \quad (x_i, y_i) \in T \tag{3}$$

where w_i are weights and (x_i, y_i) are nodes [2]. Then we find

$$I_T[f] = Q_n[f] + R_n[f] \tag{4}$$

where $R_n[f]$ is the remainder of the formula. Formula (3) has algebraic degree of precision d if

$$R_n[f] = 0 \iff I_T[f] = Q_n[f] \tag{5}$$

holds for all polynomial functions of total degree d or less (in x and in y) and a polynomial of total degree $d + 1$ exists for which (5) does not hold. In the following cubature fomula with algebraic degree of precision d is indicated as $Q_n^d[f]$; in general the degree of precision depends on the number of nodes in (3). From the wide literature for formulae (3) we refer to [3] [4] [5] and references therein. In order to obtain a numerical estimation of the error in (4) the method of embedded formulae is widely used. If $Q_n^d[f]$ is a formula of degree d and Q_{n+k}^{d+1} is a formula of degree $d + 1$, then the value

$$|Q_n^d[f] - Q_{n+k}^{d+1}[f]| \tag{6}$$

is assumed as error estimation of $Q_n^d[f]$. In [6] a method is introduced to build up pairs of cubature formulae with degree of precision $2d - 1$ and $2d + 1$ with a special feature: all the nodes (and hence all functional values) required in the calculation of formula $Q_n^d[f]$ enter the $n + k$-th order formula $Q_{n+k}^{d+1}[f]$. This kind of formulae is called embedded formulae. The advantage in using embedded formulae is that it gives the chance to increase the precision of approximation of (2) with a min-

ISSN 1390-5384

imum number of operations.

In this paper we introduce a method to build up embedded formulae which is in the line of [7] [8] [9]. These cubature formulae have a fixed number of nodes, usually the vertexes of the triangle. They require values from the derivatives of the function and their algebraic degree of precision increases with the degree of the derivatives involved. Quadrature formulae which use derivative values are classical in the literature, see [10] [11] [12]. This kind of formulae is useful when derivative values are obtained directly or indirectly: this is the case proposed in [13]. Moreover calculation of derivatives can be an easy task if it is done with the aid of symbolic calculation software[14].

The Cubature formuale introduced in this paper are achieved by integrating the expansion formulae in [1], have a fixed number of nodes and require values from derivatives in order to increase their algebraic degree of precision; moreover they have a further property: they are exact on a large class of rational functions. As far as the authors are aware this kind of exactness does not appear in the literature. The urge of cubature formulae for rational functions, as it is described, for example, in [15], motivates this paper.

The paper is organized as follows: in sec.2 we summarize previous results required in the development of the paper; among other results, we recall the expansion formula introduced in [1]. In sec.3 we introduce the class of cubature formulae; in sec.4 we investigate the remainder and its bounds. In sec.5 we search for the class of functions such that cubature formulae converge to their integral. We conclude the paper with numerical examples in sec.6.

2 Preliminaries

In this section we summarize some of the results introduced in [1], [16] and [17] in order to make this paper self explanatory.

In [16] an univariate two points expansion formula is introduced. Let $f \in C^m([0,\alpha])$: the expansion formula $B_m(\alpha)[f]$ is the $m-$th degree polynomial of the variable x

$$B(\alpha)_m[f](x) := f(0)+\sum_{i=1}^{m} S_i\left(\frac{x}{\alpha}\right)\frac{\alpha^{i-1}}{i!}\Delta_\alpha f^{(i-1)}(0) \tag{7}$$

where

$$\Delta_\alpha f(x) = f(x+\alpha) - f(x)$$

and the polynomial S_j are defined by means of the Bernoulli polynomial [18]:

$$S_j(x) = B_j(x) - B_j(0).$$

Note that expansion $B(\alpha)_m[f]$ has algebraic degree of precision:

$$p(x) \in \mathcal{P}^m([0,\alpha]) \Rightarrow p(x) = B(\alpha)_m[p](x).$$

This property is rigorously proved in [16] and it is straightforward to verify.

In [1] a bi-variate expansion formula for functions defined on the triangle is introduced; this new formula is an extension of (7) and is defined as:

$$B(\alpha,\beta)_{n,m}[f] = B(\alpha)_n \otimes B(\beta)_m[f \circ g^{-1}] \circ g \tag{8}$$

where the function g is the Duffy map [19]:

$$g(x,y) = \left(x, \frac{y}{1-x}\right) \tag{9}$$

In [1] the expansion formula (8) is introduced explicitly in the case $\alpha = 1/2$, $\beta = 1/2$; here introduce the same formula generalized for $\alpha \in (0,1)$ and $\beta \in (0,1]$.

$$\begin{aligned}
&B(\alpha,\beta)_{n,m}[f](x,y) := f(0,0)\\
&+\sum_{i=1}^{n}\frac{\alpha^{i-1}}{i!}\Delta_{(\alpha,0)} fog_{\,i-1,0}(0,0) S_i\left(\frac{x}{\alpha}\right)\\
&+\sum_{j=1}^{m}\frac{\beta^{j-1}}{j!}\Delta_{(0,\beta)} fog_{\,0,j-1}(0,0) S_j\left(\frac{y}{\beta(1-x)}\right)\\
&+\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{\alpha^{i-1}\beta^{j-1}}{i!j!}\Delta_{(\alpha,\beta)} fog_{\,i-1,j-1}(0,0) S_i\left(\frac{x}{\alpha}\right)\cdot\\
&\cdot S_j\left(\tfrac{y}{\beta(1-x)}\right)
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&\Delta_{(\alpha,0)} f(x,y) = f(x+\alpha,y) - f(x,y);\\
&\Delta_{(0,\beta)} f(x,y) = f(x,y+\beta) - f(x,y);\\
&\Delta_{(\alpha,\beta)} f(x,y) = \Delta_{(\alpha,0)}\Delta_{(0,\beta)} f(x,y) =\\
&= f(x,y) - f(x+\alpha,y) + f(x+\alpha,y+\beta) - f(x,y+\alpha)
\end{aligned}$$

Functions $fog_{k,h}$ are expansions of the derivatives of $f \circ g^{-1}$ which are involved in the eq.(8). Hence

$$fog_{k,h}(x,y) := \frac{\partial^{k+h}}{\partial x^k \partial y^h}(f \circ g^{-1})$$

And their explicit calculation results:

$$\begin{aligned}
&fog_{k,h}(x,y) = \textstyle\sum_{i=0}^{h}\frac{k!}{(k-i)!}\left(\nabla^{k+i} f \circ g^{-1}\right)\cdot\\
&\cdot\mathrm{Sym}\left(\left(\bigotimes_{l=0}^{k-i} g_x^{-1}\right)\left(\bigotimes_{l=0}^{h-i} g_y^{-1}\right)\left(\bigotimes_{l=0}^{i} g_{x\,y}^{-1}\right)\right)
\end{aligned} \tag{11}$$

Expansion $B(\alpha,\beta)_{n,m}[f]$ has properties of exactness:

Theorem 2.1 ([1]). *Expansion formula $B(\alpha,\beta)_{n+m,m}[f]$ has algebraic degree of precision, it is exact on polynomial of degree (n,m) (see fig.1).*

Expansion $B(\alpha,\beta)_{n,m}[f]$ is exact on rational functions $r(x,y) = p(x, y/(1-x))$ where $p(x,y)$ is a polynomial of degree (n,m).

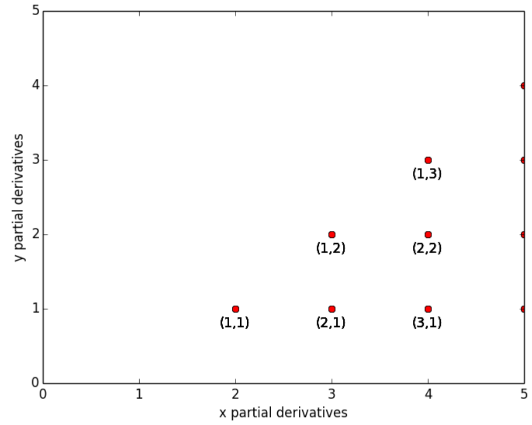

Figure 1: The algebraic degree of precision (n, m) plotted in the plane ($x-$partial derivatives, $y-$partial derivatives).

Proof. Let us consider the monomial $p = x^n y^m$: it is the polynomial with highest degree among all of degree (n, m). Proving that $B(\alpha, \beta)_{n+m,m}[p] = p$ will ensure the precision of the expansion.

We note that $p \circ g^{-1} = x^n \big(y(1-x)\big)^m$ is a polynomial of degree $(n+m, m)$; because of the precision of $B_n(\alpha)[f]$ and the properties of tensor product, it follows:

$$B_{n+m}(\alpha) \otimes B_m(\beta)[p \circ g^{-1}] = p \circ g^{-1} \tag{12}$$

then, recalling the definition in eq.(8), we found:

$$\begin{aligned} &B(\alpha, \beta)_{n+m,m}[p] = B_n(\alpha) \otimes B_m(\beta)[p \circ g^{-1}] \circ g = \\ &= p \circ g^{-1} \circ g = p. \end{aligned} \tag{13}$$

The same argument can be used to prove the exactness of the expansion on rational functions. It is sufficient to note that if $p = x^n y^m$, then $r \circ g^{-1} = x^n \left(\frac{y}{1-x}(1-x)\right)^m = x^n y^m$. Then the result follows analogoulsy of eqs.(12,13). □

3 The new cubature formula

Theorem 3.1. *Let f be a sufficiently smooth function defined on the standard triangle $T = \{(x, y) : x > 0 \quad x + y \leq 1\}$. Let be $\alpha \in (0, 1)$ and $\beta \in (0, 1]$.*

The cubature formula

$$\begin{aligned} &C(\alpha, \beta)_{n+m,m}[f] := \frac{1}{2} f(0, 0) \\ &+ \sum_{i=1}^{n+m} \frac{\alpha^{i-1}}{i!} C^i_\alpha \, \Delta_{(\alpha,0)} fog_{\,i-1,0}(0, 0) \\ &+ \sum_{j=1}^{m} \frac{\beta^{j-1}}{j!\,2} C^j_\beta \, \Delta_{(0,\beta)} fog_{\,0,j-1}(0, 0) \\ &+ \sum_{i=1}^{n+m} \sum_{j=1}^{m} \frac{\alpha^{i-1}\beta^{j-1}}{i!j!} C^i_\alpha \, C^j_\beta \, \Delta_{(\alpha,\beta)} fog_{\,i-1,j-1}(0, 0) \end{aligned} \tag{14}$$

where

$$\begin{aligned} &C^i_\alpha = \tfrac{\alpha^2}{(i+1)(i+2)} S_{i+2}\left(\tfrac{1}{\alpha}\right) + \tfrac{\alpha}{i+1} \cdot \\ &\cdot B_{i+1}(0) - \tfrac{1}{2} B_i(0) \end{aligned} \tag{15}$$

$$C^j_\beta = \frac{\beta}{j+1} S_{j+1}\left(\frac{1}{\beta}\right) - B_j(0), \tag{16}$$

and $fog_{k,h}(x, y)$ is in eq.(11), has algebraic degree of precision (n, m).

The cubature formula $C(\alpha, \beta)_{n,m}[f]$ is exact on rational functions $r(x, y) = p(x, y/(1-x))$ where $p(x, y)$ is a polynomial of degree (n, m).

Proof. Cubature formulae (14) are achieved by integrating expansion formulae $B(\alpha, \beta)_{n+m,m}$.

$$\int_0^1 \int_0^{1-x} B(\alpha)_{n+m} \otimes B(\beta)_m [f \circ g^{-1}] \circ g(x, y) \, \mathrm{d}x \, \mathrm{d}y$$

we change the variables to integrate on the square: the function g^{-1} maps the triangle on the square and $J_{g^{-1}} = (1-x)$ is its Jacobian. It result

$$\begin{aligned} &\textstyle\int_0^1 \int_0^1 B(\alpha)_{n+m} \otimes B(\beta)_m [f \circ g^{-1}] \circ g \circ g^{-1}(x, y) J_{g^{-1}} \, \mathrm{d}x \, \mathrm{d}y = \\ &\textstyle\int_0^1 \int_0^1 B(\alpha)_{n+m} \otimes B(\beta)_m [f \circ g^{-1}](x, y) J_{g^{-1}} \, \mathrm{d}x \, \mathrm{d}y \end{aligned}$$

When we breakdown the function in the integral, to obtain the cubature formula we need to integrate the following:

$$C^i_\alpha \;:=\; \textstyle\int_0^1 \int_0^1 S_i\left(\frac{x}{\alpha}\right)(1-x) \, \mathrm{d}x \, \mathrm{d}y \tag{17}$$

$$\tfrac{1}{2} C^j_\beta \;:=\; \textstyle\int_0^1 \int_0^1 S_j\left(\frac{y}{\beta}\right)(1-x) \, \mathrm{d}x \, \mathrm{d}y \tag{18}$$

$$C^i_\alpha \, C^j_\beta \;=\; \textstyle\int_0^1 \int_0^1 S_j\left(\frac{y}{\beta}\right) S_i\left(\frac{x}{\alpha}\right)(1-x) \, \mathrm{d}x \, \mathrm{d}y \tag{19}$$

We note that the rational functions $S_j\big(y/\big(\beta(1-x)\big)\big)$ now are polynomial in y thanks to the change of variables (*see eqs.8-10*). Because of the separation of variables, the values C^i_α and C^j_β in eqs.(17,18) can be rewritten as:

$$C^i_\alpha = \int_0^1 S_i\left(\frac{x}{\alpha}\right)(1-x) \, \mathrm{d}x \qquad C^j_\beta = \int_0^1 S_j\left(\frac{y}{\beta}\right) \mathrm{d}y$$

and for the same reason the equality in eq.(19) holds.

To conclude the integration we note that obtaining both C^i_α and C^j_β is straightforward once that we have:

$$\begin{aligned} &\textstyle\int S_i\left(\frac{x}{\alpha}\right) \mathrm{d}x = \int \Big(B_i\left(\frac{x}{\alpha}\right) - B_i(0)\Big) \mathrm{d}x = \frac{\alpha}{i+1} B_{i+1}\left(\frac{x}{\alpha}\right) - \\ &+ B_i(0)\, x + c \end{aligned}$$

which follows from the integral property of Bernoulli polynomials [18].

In sec 2 it is recalled that the expansion formula $B(\alpha,\beta)[f]$ is exact on polynomial and on a class of rational functions. Cubature formula $C(\alpha,\beta)[f]$ inherit the same exactness. □

In the formula above α is strictly bounded by 1 because Duffy's map (9) is continuous and invertible in the triangle except for the point $(1,0)\in\mathbb{R}^2$, which is involved when $\alpha=1$. On the other hand, when it is applied in the calculation of integrals, the singularity in $(1,0)\in\mathbb{R}^2$ disappears and it is possible to define a cubature formula with the vertexes of the triangle as nodes. Formula $C(1,1)_{n,m}[f]$ comes as limit case of $C(\alpha,1)_{n,m}[f]$:

$$\begin{array}{rcl} C(1,1)_{n,m}[f] & = & \frac{1}{2}f(0,0)+ \\ & + & \sum_{i=1}^{n}\frac{1}{i!}\left(-\frac{B_i}{2}-\frac{B_{i+1}}{(1+i)}+\frac{S_{i+2}(1)}{(2+i)(1+i)}\right)\cdot \\ & & \cdot\Delta_{(1,0)}fog_{\,i-1,0}(0,0) \\ & + & \sum_{j=1}^{m}\frac{1}{j!2}\left(-B_j+\frac{1}{(j+1)}S_{j+1}(1)\right)\cdot \\ & & \cdot\Delta_{(0,1)}fog_{\,0,j-1}(0,0) \\ & + & \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{i!j!}\left(-\frac{B_i}{2}-\frac{B_{i+1}}{(1+i)}+\right. \\ & & \left.+\frac{S_{i+2}(1)}{(2+i)(1+i)}\right)\left(-B_j+\frac{1}{2(j+1)}S_{j+1}(1)\right) \\ & \cdot & \Delta_{(1,1)}fog_{\,i-1,j-1}(0,0) \end{array}$$

Cubature formula $C(\alpha,\beta)_{n,m}[f]$ can be written in terms of f and its derivatives, without the use of the function fog; if this is the case, after rearrangements, the cubature formula becomes:

$$\begin{array}{rcl} C(\alpha,\beta)_{n,m}[f] & = & \sum_{i=1}^{4}\sum_{h=0}^{n+1}\sum_{k=0}^{m+1}A_{i,h,k}f^{(h,k)}(x_i,y_i)= \\ & = & C(\alpha,\beta)_{n-1,m-1}+ \\ & & +\sum_{i=1}^{4}\Big(\sum_{h=0}^{n}A_{i,h,n+1}f^{(h,m+1)}(x_i,y_i)+ \\ & & +\sum_{k=0}^{m}A_{i,m+1,k}f^{(n+1,k)}(x_i,y_i) \\ & & +\sum_{h=0}^{n}\sum_{k=0}^{m}A_{i,n+1,m+1}f^{(n+1,m+1)}(x_i,y_i)\Big) \end{array} \tag{20}$$

where (x_i,y_i) are the four nodes, $A_{i,h,k}$ respectively are the weights and both (x_i,y_i) and $A_{i,h,k}$ depend on α and β.

We note that if $f=1$ then $C(\alpha,\beta)_{n,m}[f]=1/2$ because of the precision of the formula: thus formula (20) gives:

$$A_{1,0,0}+A_{2,0,0}+A_{3,0,0}+A_{4,0,0}=1/2$$

where all other $A_{i,h,k}$ are multiplied by 0. Hence the sum of the weights related to the functional values is the area of the triangle.

Remark 3.2. We point out that cubature formula of degree (n,m) contains cubature formula of degree $(n-1,m-1)$ as in (20). This ensures that (20) generates a couple of embedded formulae.

4 The remainder formula

In this section we introduce the remainder formula and its bounds. The reminder is achieved by the use of the Sard's kernels [20].

Theorem 4.1. *We let*

$$R(\alpha,\beta)_{n,n}[f]=\int_0^1\int_0^{1-x}f\,dx\,dy-C(\alpha,\beta)_{n,n}[f].$$

then we find:

$$\begin{array}{rcl} R(\alpha,\beta)_{n,n}[f] & = & \frac{1}{k!}\sum_{i=0}^{n}\binom{k}{i}\Big(\int_0^1 K_1^i(u)f^{(k+1-i,i)}(u,0)\,du+ \\ & & +\int_0^1 K_2^i(v)f^{(i,k+1-i)}(0,v)\,dv\Big)+ \\ & & +\begin{cases} \frac{1}{n!n!}\int_0^1\int_0^{1-v}K_{n,n}(u,v)\cdot & \\ \cdot f^{(n+1,n+1)}(u,v)\,du\,dv & k=2n+1 \\ \frac{1}{2(n-1)!n!}\Big(\int_0^1\int_0^{1-v}K_{n,n-1}(u,v)\cdot & \\ \cdot f^{(n+1,n)}(u,v)\,du\,dv & \\ +\int_0^1\int_0^{1-v}K_{n-1,n}(u,v)\cdot & \\ \cdot f^{(n,n+1)}(u,v)\,du\,dv\Big) & k=2n \end{cases} \end{array} \tag{21}$$

where

$$\begin{array}{rcl} K_1^i(u) & := & \int_0^1\int_0^{1-x}y^i(x-u)_+^{k-i}\,dx\,dy-C(\alpha,\beta)_{i,i}\cdot \\ & & \cdot[y^i(x-u)_+^{k-i}] \\ K_2^i(v) & := & \int_0^1\int_0^{1-x}x^i(y-v)_+^{k-i}\,dx\,dy-C(\alpha,\beta)_{i,i}\cdot \\ & & \cdot[x^i(y-v)_+^{k-i}] \\ K_{n,n}(u,v) & := & \int_0^1\int_0^{1-x}(x-u)_+^n(y-v)_-^n\,dx\,dy- \\ & & +C(\alpha,\beta)_{n,n}[(x-u)_+^n(y-v)_-^n] \\ K_{n,n-1}(u,v) & := & \int_0^1\int_0^{1-x}(x-u)_+^n(y-v)_-^{n-1}\,dx\,dy- \\ & & +C(\alpha,\beta)_{n,n}[(x-u)_+^n(y-v)_-^{n-1}] \end{array}$$

Proof. The formula (21) comes from a straightforward application of Sard's kernels theorem [20]. □

If we let

$$M_{i,j}=sup|f^{(i,j)}(x,y)|$$

we find the following estimation for the remainder

$$\begin{array}{l} \left|R(\alpha,\beta)_{n,n}[f]\right|\le\frac{1}{k!}\sum_{i=0}^{n}\binom{k}{i}\Big(M_{k+1-i,i}\int_0^1\left|K_1^i(u)\right|du+ \\ \quad +M_{i,k+1-i}\int_0^1\left|K_2^i(v)\,dv\right|\Big)+ \\ \quad +\begin{cases} \frac{1}{n!n!}M_{n+1,n+1}\int_0^1\int_0^{1-v}|K_{n,n}(u,v)|\,du\,dv & k=2n+1 \\ \frac{1}{2(n-1)!n!}\cdot & \\ \cdot\Big(M_{n+1,n}\int_0^1\int_0^{1-v}|K_{n,n-1}(u,v)|\,du\,dv+ & \\ +M_{n,n+1}\int_0^1\int_0^{1-v}|K_{n-1,n}(u,v)|\,du\,dv\Big) & k=2n \end{cases} \end{array}$$

5 Convergence theorems

In this section we investigate the convergence of the cubature formula; in particular we prove the following:

Theorem 5.1. *Let f be a function defined on the triangle. If $f \in C^\infty$ and*

$$\|\frac{\partial^{h+k}}{\partial x^k \partial y^h} f(x,y)\|_\infty < M \quad h,k = 0,\ldots \tag{22}$$

then cubature formula $C(\alpha,\beta)_{n+m,m}[f]$ converges to the integral:

$$\lim_{\substack{n\to\infty \\ m\to\infty}} C(\alpha,\beta)_{n+m,m} = \int_0^1 \int_0^{1-x} f(x,y)\, dx\, dy$$

Proof. We begin the proof stating an uniform bound in n, m for $C(\alpha,\beta)_{n,m}$. First we have:

$$\begin{aligned} \Big|C(\alpha,\beta)_{n,m}[f]\Big| \leq M \Bigg| \tfrac{1}{2} + \sum_{i=1}^{n} \frac{\alpha^{i-1}}{i!} C_\alpha^i + \\ + \sum_{j=1}^{m} \frac{\beta^{j-1}}{j!2} C_\beta^j + \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\alpha^{i-1}\beta^{j-1}}{i!j!} C_\alpha^i C_\beta^j \Bigg| \end{aligned} \tag{23}$$

and hence we have to prove the convergence of the three series in eq.(23). Since the method is the same for all of them, let us focus on the series:

$$\sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{n!} |C_\alpha^n|$$

Taking into account the definition of C_α^i in eq.(15), the convergence of the series above depends on the convergence of:

$$\sum_{n}^{\infty} \frac{\alpha^{n-1}}{n!} |B_n(0)| \qquad \text{and} \qquad \sum_{n}^{\infty} \frac{\alpha^{n-1}}{n!} \left| B_n\left(\frac{1}{\alpha}\right)\right|. \tag{24}$$

The second series in eq.(24) can be related to the first series in eq.(24) by using the root test and the equality

$$B_n\left(\frac{1}{\alpha}\right) = \left(B + \frac{1}{\alpha}\right)^n$$

where $B^i = B_i(0)$, stated in [18]; in fact we have:

$$\lim_{n\to\infty} \left| B + \frac{1}{\alpha}\right|^n = \lim_{n\to\infty} |B_n(0)| + \left|\frac{1}{\alpha}\right|^n.$$

Now let us focus on the first series in eq.(24). For that we take into account another bound stated in [18]:

$$\frac{|B_n(0)|}{n!} < \frac{1}{(2\pi)^{n-2}(n/2)!} \tag{25}$$

Hence the first series in eq.(24) is bounded by:

$$\sum_{n}^{\infty} \frac{\alpha^{n-1}}{n!} |B_n(0)| \leq \sum_{n}^{\infty} \frac{\alpha^{n-1}}{(2\pi)^{n-2}(n/2)!} \tag{26}$$

which converges by test root.

We have proven that the inquality (23) can be uniformly bounded, in n and m, by using a certain value S:

$$\Big|C(\alpha,\beta)_{n,m}[f]\Big| \leq MS \tag{27}$$

To conclude the proof we need:

$$\left|C(\alpha,\beta)_{n+m,m}[f] - \int_0^1 \int_0^{1-x} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\right| < \varepsilon \tag{28}$$

Because of the density of polynomials in C^∞ we have a polynomial $p(x,y)$ s.t.

$$\begin{gathered} \left|\tfrac{\partial^{h+k}}{\partial x^k \partial y^h}\Big(f(x,y) - p(x,y)\Big)\right| < \varepsilon \\ h = 1,\ldots,n \quad k = 1,\ldots,m \end{gathered}$$

The degree of $p(x,y)$ will be (r,t). If $r \leq n$ and $t \leq m$ we find that

$$C(\alpha,\beta)_{n+m,m}[p] = \int_0^1 \int_0^{1-x} p(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{29}$$

and a straightforward calculation yields:

$$\begin{aligned} & |C(\alpha,\beta)_{n+m,m}[f] - C(\alpha,\beta)_{n+m,m}[p] + \\ & +C(\alpha,\beta)_{n+m,m}[p] - \textstyle\int_0^1\int_0^{1-x} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \leq \\ \leq\ & |C(\alpha,\beta)_{n+m,m}[f-p] - \\ & + \textstyle\int_0^1\int_0^{1-x} \big(f(x,y) - p(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y\Big| \leq \\ \leq\ & \Big|C(\alpha,\beta)_{n+m,m}[f-p]\Big| + \\ & + \textstyle\int_0^1\int_0^{1-x} \big|f(x,y) - p(x,y)\big|\,\mathrm{d}x\,\mathrm{d}y \leq S\varepsilon + 1/2\varepsilon \end{aligned}$$

If $r > n$ or $t > m$ the condition (29) does not hold and the proof need a slight modification: we note that

$$C(\alpha,\beta)_{r+t,t}[p] = \int_0^1 \int_0^{1-x} p(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

and hence:

$$\begin{aligned} & \Big|C(\alpha,\beta)_{n+m,m}[f] - C(\alpha,\beta)_{r+t,t}[f]\Big| \\ +\ & |C(\alpha,\beta)_{r+t,t}[f] - C(\alpha,\beta)_{r+t,t}[p] + \\ & + C(\alpha,\beta)_{r+t,t}[p] - \textstyle\int_0^1\int_0^{1-x} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\Big| \leq \\ \leq\ & MS\varepsilon + |C(\alpha,\beta)_{r+t,t}[f-p] - \\ & + \textstyle\int_0^1\int_0^{1-x} \big(f(x,y) - p(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y\Big| \leq \\ \leq\ & MS\varepsilon + \Big|C(\alpha,\beta)_{r+t,t}[f-p]\Big| + \\ & + \textstyle\int_0^1\int_0^{1-x} \big|f(x,y) - p(x,y)\big|\,\mathrm{d}x\,\mathrm{d}y \\ \leq\ & MS\varepsilon + S\varepsilon + 1/2\varepsilon \end{aligned}$$

where, because of (27), we let

$$\Big|C(\alpha,\beta)_{n+m,m}[f] - C(\alpha,\beta)_{r+r,t}[f]\Big| \leq MS\varepsilon$$

□

Theorem 5.1 can be further improved because the condition (22) is not sharp with respect to the convergence of the cubature in (14).

Corollary 5.2. *In thm5.1 the condition (22) can be replaced by the following:*

$$\| \frac{1}{(2\pi)^h(2\pi)^k} \frac{\partial^{h+k}}{\partial x^k \partial y^h} f(x,y) \|_\infty \leq M$$

proof. The new bound of the partial derivatives of f affects the series in eq.(24); in particular the first one will be:

$$\sum_n^\infty \frac{(2\pi)^{n-2}\alpha^{n-1}}{n!} |B_n(0)|$$

however the bound for $B_n(0)$ in eq.(25) is still sufficient to bound the series above with a convergent one as in eq.(26).□

We conclude the section with the following remark. The real part $f(x)$ of a complex function such that

$$|f(z)| \leq M \exp(a\pi|z|).$$

is an *exponential type function*: for such kind of functions we have

$$\| \frac{1}{(a\pi)^h} \frac{d^h}{dx^k} f(x) \|_\infty \leq M.$$

Hence, condition (22) can be rephrased requiring that function $f(x,y)$ is a tensor product of exponential type functions.

6 Numerical Examples

In these numerical tests we fix the value of n (which is the increasing value from 1 to 8 in the first column) and we report on the difference between $C(\alpha,\beta)_{n,n}[f]$ and the exact integral, we also report the value of

$$|C(\alpha,\beta)_{n,n}[f] - C(\alpha,\beta)_{n+1,n+1}[f]|;$$

these are the two values in the braces. We report the results for $(\alpha,\beta) = (1/2,1/2), (1/3,1/2), (1/2,1), (1,1)$.

In this series of numerical test the functions are the same used in [8]: the results are comparable.

$$f(x,y) = \sin(\pi/4 \cdot x + \pi/6 \cdot y)$$

n	$(1/2, 1/2)$	$(1/3, 1/2)$
1	$\{-1.19 * 10^{-3}, -6.68 * 10^{-3}\}$	$\{-2.70 * 10^{-3}, -1.89 * 10^{-3}\}$
2	$\{-5.27 * 10^{-4}, -5.38 * 10^{-4}\}$	$\{-8.0 * 10^{-4}, -8.23 * 10^{-4}\}$
3	$\{1.15 * 10^{-5}, 1.01 * 10^{-5}\}$	$\{1.91 * 10^{-5}, 1.55 * 10^{-5}\}$
4	$\{1.43 * 10^{-6}, 1.45 * 10^{-6}\}$	$\{3.65 * 10^{-6}, 3.71 * 10^{-6}\}$
5	$\{-1.50 * 10^{-8}, -1.23 * 10^{-8}\}$	$\{-5.94 * 10^{-8}, -4.77 * 10^{-8}\}$
6	$\{-2.68 * 10^{-9}, -2.62 * 10^{-9}\}$	$\{-1.17 * 10^{-8}, -1.18 * 10^{-8}\}$
7	$\{-6.53 * 10^{-11}, 3.17 * 10^{-11}\}$	$\{4.64 * 10^{-11}, 1.18 * 10^{-10}\}$
8	$\{-9.68 * 10^{-11}, 2.58 * 10^{-12}\}$	$\{-7.24 * 10^{-11}, 2.72 * 10^{-11}\}$

n	$(1/2, 1)$	$(1, 1)$
1	$\{1.70 * 10^{-3}, 1.92 * 10^{-3}\}$	$\{7.42 * 10^{-3}, 7.60 * 10^{-3}\}$
2	$\{-2.23 * 10^{-4}, -2.35 * 10^{-4}\}$	$\{-1.79 * 10^{-4}, -2.18 * 10^{-4}\}$
3	$\{1.25 * 10^{-5}, 1.17 * 10^{-5}\}$	$\{3.91 * 10^{-5}, 4.09 * 10^{-5}\}$
4	$\{7.51 * 10^{-7}, 7.33 * 10^{-7}\}$	$\{-1.83 * 10^{-6}, -2.17 * 10^{-6}\}$
5	$\{1.84 * 10^{-8}, 2.01 * 10^{-8}\}$	$\{3.42 * 10^{-7}, 3.64 * 10^{-7}\}$
6	$\{-1.64 * 10^{-9}, -1.72 * 10^{-9}\}$	$\{-2.14 * 10^{-8}, -2.54 * 10^{-8}\}$
7	$\{7.20 * 10^{-11}, 1.68 * 10^{-10}\}$	$\{4.01 * 10^{-9}, 4.37 * 10^{-9}\}$
8	$\{-9.68 * 10^{-11}, 1.86 * 10^{-12}\}$	$\{-3.69 * 10^{-10}, -3.15 * 10^{-10}\}$

$$f(x,y) = \sinh(\pi/4 \cdot x + \pi/6 \cdot y)$$

n	$(1/2, 1/2)$	$(1/3, 1/2)$
1	$\{1.29 * 10^{-3}, 7.02 * 10^{-3}\}$	$\{2.89 * 10^{-3}, 1.99 * 10^{-3}\}$
2	$\{5.95 * 10^{-4}, 5.83 * 10^{-4}\}$	$\{8.99 * 10^{-4}, 8.79 * 10^{-4}\}$
3	$\{1.20 * 10^{-5}, 1.04 * 10^{5}\}$	$\{1.99 * 10^{-5}, 1.58 * 10^{-5}\}$
4	$\{1.58 * 10^{-6}, 1.56 * 10^{-6}\}$	$\{4.01 * 10^{-6}, 3.95 * 10^{-6}\}$
5	$\{1.55 * 10^{-8}, 1.27 * 10^{-8}\}$	$\{6.13 * 10^{-8}, 4.88 * 10^{-8}\}$
6	$\{2.72 * 10^{-9}, 2.81 * 10^{-9}\}$	$\{1.24 * 10^{-8}, 1.24 * 10^{-8}\}$
7	$\{-9.35 * 10^{-11}, 3.27 * 10^{-11}\}$	$\{2.13 * 10^{-11}, 1.21 * 10^{-10}\}$
8	$\{-1.26 * 10^{-10}, 2.76 * 10^{-12}\}$	$\{-1.01 * 10^{-10}, 2.85 * 10^{-11}\}$

n	$(1/2, 1)$	$(1, 1)$
1	$\{-1.78 * 10^{-3}, -2.06 * 10^{-3}\}$	$\{-8.04 * 10^{-3}, -8.36 * 10^{-3}\}$
2	$\{2.76 * 10^{-4}, 2.62 * 10^{-4}\}$	$\{3.24 * 10^{-4}, 2.82 * 10^{-4}\}$
3	$\{1.31 * 10^{-5}, 1.23 * 10^{-5}\}$	$\{4.21 * 10^{-5}, 4.52 * 10^{-5}\}$
4	$\{7.90 * 10^{-7}, 8.09 * 10^{-7}\}$	$\{-3.14 * 10^{-6}, -2.77 * 10^{-6}\}$
5	$\{-1.93 * 10^{-8}, -2.12 * 10^{-8}\}$	$\{-3.69 * 10^{-7}, -4.05 * 10^{-7}\}$
6	$\{1.90 * 10^{-9}, 1.85 * 10^{-9}\}$	$\{3.61 * 10^{-8}, 3.19 * 10^{-8}\}$
7	$\{4.95 * 10^{-11}, 1.77 * 10^{-10}\}$	$\{4.24 * 10^{-9}, 4.82 * 10^{-9}\}$
8	$\{-1.27 * 10^{-10}, 2.03 * 10^{-12}\}$	$\{-5.77 * 10^{-10}, -3.99 * 10^{-10}\}$

$$f(x,y) = \cos(\sqrt{1+x^2+y^2})$$

n	$(1/2, 1/2)$	$(1/3, 1/2)$
1	$\{-7.49 * 10^{-3}, -8.34 * 10^{-3}\}$	$\{-2.14 * 10^{-2}, -2.22 * 10^{-2}\}$
2	$\{8.53 * 10^{-4}, 3.83 * 10^{-4}\}$	$\{8.09 * 10^{-4}, 4.428 * 10^{-}6\}$
3	$\{4.69 * 10^{-4}, 4.74 * 10^{-4}\}$	$\{8.05 * 10^{-4}, 8.05 * 10^{-4}\}$
4	$\{-4.57 * 10^{-6}, -2.19 * 10^{-6}\}$	$\{1.87 * 10^{-7}, 1.06 * 10^{-5}\}$
5	$\{-2.37 * 10^{-6}, -2.39 * 10^{-6}\}$	$\{-1.04 * 10^{-5}, -1.03 * 10^{-5}\}$
6	$\{2.04 * 10^{-8}, -9.64 * 10^{-10}\}$	$\{-1.07 * 10^{-7}, -2.03 * 10^{-7}\}$
7	$\{2.14 * 10^{-8}, 2.08 * 10^{-8}\}$	$\{9.54 * 10^{-8}, 9.35 * 10^{-8}\}$

n	$(1/2, 1)$	$(1, 1)$
1	$\{1.54 * 10^{-2}, 1.50 * 10^{-2}\}$	$\{6.08 * 10^{-2}, 5.91 * 10^{-2}\}$
2	$\{4.07 * 10^{-4}, -5.41 * 10^{-5}\}$	$\{1.72 * 10^{-3}, 4.69 * 10^{-4}\}$
3	$\{4.61 * 10^{-4}, 4.57 * 10^{-4}\}$	$\{1.25 * 10^{-3}, 1.19 * 10^{-3}\}$
4	$\{3.88 * 10^{-6}, 1.70 * 10^{-6}\}$	$\{5.63 * 10^{-5}, 1.37 * 10^{-5}\}$
5	$\{2.17 * 10^{-6}, 2.11 * 10^{-6}\}$	$\{4.25 * 10^{-5}, 4.04 * 10^{-5}\}$
6	$\{6.66 * 10^{-8}, -1.94 * 10^{-8}\}$	$\{2.08 * 10^{-6}, 4.54 * 10^{-7}\}$
7	$\{8.61 * 10^{-8}, 8.40 * 10^{-8}\}$	$\{1.63 * 10^{-6}, 1.55 * 10^{-6}\}$

We recall that cubature formulae are exact on rational function $r(x,y) = q(x, y/(1-x))$ where $q(x,y)$ is a polynomial. In particular they are exact on the functions:

$$f(x,y) = (p(x) + q(x)y)/(1-x)$$

where $p(x)$ and $q(x)$ are polynomials. This is a subset of the class of functions investigated in [15]. The following examples test cubature formulae on rational functions with a bivariate polynomial numerator and linear denominator [15].

In this case we introduce the results only for the cases $(\alpha,\beta) = (1/2,1/2), (1/3,1/2), (1/2,1)$

$$f(x,y) = y^8/(1-x)$$

n	$(1/2, 1/2)$	$(1/3, 1/2)$
1	$\{0.0116845, 0.000881619\}$	$\{0.0122314, 0.00257745\}$
2	$\{0.0108028, -4.74718 * 10^{-6}\}$	$\{0.00965392, -0.00338817\}$
3	$\{0.0108076, 0.00550364\}$	$\{0.0130421, 0.0102792\}$
4	$\{0.00530395, 0.00213346\}$	$\{0.00276287, -0.0014947\}$
5	$\{0.00317049, 0.00209731\}$	$\{0.00425757, 0.0033233\}$
6	$\{0.00107318, 0.000841697\}$	$\{0.000934277, 0.000702796\}$
7	$\{0.000231481, 0.000231481\}$	$\{0.000231481, 0.000231481\}$
8	$\{1.6985 * 10^{-11}, 0.\}$	$\{1.6985 * 10^{-11}, 0.\}$

n	$(1/2, 1)$
1	$\{-0.0722897, -0.112847\}$
2	$\{0.0405575, 0.0188368\}$
3	$\{0.0217207, 0.0467578\}$
4	$\{-0.0250371, -0.00455729\}$
5	$\{-0.0204798, -0.0241633\}$
6	$\{0.00368345, -0.0000202546\}$
7	$\{0.0037037, 0.0037037\}$
8	$\{1.6985 * 10^{-11}, 0.\}$

$$f(x, y) = (xy)^4/(1 - x)$$

n	$(1/2, 1/2)$	$(1/3, 1/2)$
1	$\{0.0001547, 0.000108507\}$	$\{0.000203148, 0.000195283\}$
2	$\{0.000046193, 0.0000759549\}$	$\{7.86574 * 10^{-6}, 0.000258154\}$
3	$\{-0.0000297619, -8.68056 * 10^{-6}\}$	$\{-0.000250289, -0.000177945\}$
4	$\{-0.0000210813, -0.000062004\}$	$\{-0.0000723431, -0.0007697\}$
5	$\{0.0000409226, 0.000093006\}$	$\{0.000697357, 0.00102391\}$
6	$\{-0.0000520833, -0.0000520833\}$	$\{-0.000326551, -0.000326551\}$
7	$\{-6.29467 * 10^{-13}, 0.\}$	$\{-6.29467 * 10^{-13}, 0.\}$

n	$(1/2, 1)$
1	$\{-0.000984623, -0.000868056\}$
2	$\{-0.000116567, -0.0000868056\}$
3	$\{-0.0000297619, -8.68056 * 10^{-6}\}$
4	$\{-0.0000210813, -0.000062004\}$
5	$\{0.0000409226, 0.000093006\}$
6	$\{-0.0000520833, -0.0000520833\}$
7	$\{-6.29467 * 10^{-13}, 0.\}$

References

[1] Costabile, F.; Guzzardi, L. 2009. “Extending univariate expansion formulae on the simplex”. *IJNMA*, 1:1–17.

[2] Stroud., A. 1971. “Approximate calculation of multiple integrals”. *Prentice-Hall, Inc*.

[3] Lyness, J.; Cools, R. 1993. “A survey of numerical cubature over triangles”. *In A half-century of computational mathematics, Proc. Sympos. Appl. Math*, 48:1943–1993.

[4] Cools, R.; Rabinowitz, P. 1993. “Monomial Cubature Rules since Stroud: a compilation”. *J. Comput. Appl. Math*, 48:309–326.

[5] Cools, R. 2003. “An Encyclopedia of Cubature Formula”. *J. Complexity*.

[6] Cools, R.; Haegemans, A. 1989. “On the construction of multi-dimensional embedded cubature formulae”. *Numer. Math*, 55:735–745.

[7] Costabile, F.; Dell’Accio, F. 2001. “Expansions over a simplex of real functions by means of Bernoulli polynomials”. *Num. Alg*, 28:63–86.

[8] Costabile, F.; Dell’Accio, F. 2007. “New embedded boundary-type quadrature formulas for the simplex”. *Numer Algor*, 45:253–267.

[9] Costabile, F.; Dell’Accio, F.; Guzzardi, L. 2008. “New bi-variate polynomial expansions with boundary data on the simplex”. *Calcolo*, 45:177–192.

[10] Squire, W. 1961. “Some applications of quadrature by differentiation”. *J. Soc. Ind. Appl. Math*, 9:94–108.

[11] Varma, A. 1986. “On Birkhoff quadrature formulas”. *Proc. Amer. Math. Soc*, 97:38–40.

[12] Costabile, F.; Longo, E. 2010. “A Birkhoff interpolation problem and application”. *Calcolo*, 47:49–63.

[13] Kim, K.; Cools, R.; Ixaru, L. 2002. “Quadrature rules using first derivatives for oscillatory integrands”. *J. Comput. Appl. Math*, 140:479–497.

[14] Rajasekaran, S. 2007. “Symbolic computation and differential quadrature method - A boon to engineering analysis”. *Struct. Eng. Mech*, 27:713–739.

[15] Rathod, H.; Islam, M. 1998. “Integration of rational function of bi-variate polynomial numerators with linear denominator over a (-1,1) square in a local parametric two-dimensional space”. *Comput. Methods Appl. Mech. Engrg*, 161:195–213.

[16] Costabile, F. 1999. “Expansions of real functions in Bernoulli polynomials and applications”. *Conf. Sem., University Bari*, 273.

[17] Costabile, F. F.and Dell’Accio. 2001. “Expansion over a rectangle of real functions in Bernoulli polynomials and applications”. *BIT*, 41:451.

[18] *http://mathworld.wolfram.com/bernoullipolynomial.html*.

[19] Duffy, M. 1982. “Quadrature over a pyramid or cube of integrands with a singularity at a vertex”. *SIAM J. Numer. Anal.*, 19:1260–1262.

[20] Engels, H. 1980. “Numerical Quadrature and Cubature”. *Academic Press*.