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The Fundamental Group of a Group Acting on a Topological Space
El Grupo Fundamental de un Grupo que Actúa en un Espacio Topológico

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Abstract

In 1966, F. Rhodes introduced the idea of the fundamental group of a group acting on a topological space. His article contains summarized proofs of results and has been studied since then primarily because the category of transformation groups is more general than the category of topological spaces. In this article, a thorough study of Rhodes's work is presented, providing examples to enrich the theory. Dr. James Montaldi from the University of Manchester has recently provided a more general and applicable approach Rhodes's main theorem. His results are also analyzed here.

Keywords. Algebraic Topology, Fundamental Group, Group Action

Resumen

En 1966, F. Rhodes introdujo la idea del grupo fundamental de un grupo G de homeomorfismos de un espacio topológico X. Su artículo contiene demostraciones que resumen los resultados importantes y ha sido estudiado desde entonces, principalmente debido a que la categoría de grupo de transformación es más general que la categoría de los espacios topológicos. En este artículo, un estudio a fondo del trabajo de Rhodes se presenta con ejemplos para enriquecer la teoría. El Dr. James Montaldi de la Universidad de Manchester ha contribuido recientemente a esta teoría con una forma más general y aplicable del teorema principal de Rhodes. Sus resultados también se analizan aquí.

Palabras Clave. Topología Algebraica, Grupo Fundamental, Acción de Grupo

Introduction

The fundamental group of a transformation group (X, G) of a group G acting on a topological space X generalizes the notion of the ordinary fundamental group $\pi_1(X, x_0)$ of X by incorporating the action of G on X. We will discuss in detail some of the results presented by F. Rhodes [1] in his article "On the Fundamental Group of a Transformation Group." Rhodes' main result deals with the situation in which the structure of the fundamental group of (X, G) is determined by the structure of $\pi_1(X, x_0)$ together with an appropriate action of G on $\pi_1(X, x_0)$. We shall also illustrate the general theory using well-know actions on topological spaces: Euclidean space, regular polygons, spheres, and the torus, on which the groups of integers, orthogonal matrices, and cyclic groups act.

The objective is to provide details in the proofs as well as to supplement the theory with concrete examples.

Groups Acting on Topological Spaces

Let G be a group and X be a topological space. We refer to reader to [2] and [3] for the relevant definitions.

We call the pair (X, G) a *transformation group* if G *acts continuously on* X in the sense below.

Definition 1. A *group action* of a group G on a set X is a map

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g \cdot x \end{aligned}$$

satisfying the following:

1. For each $g \in G$, the map $x \mapsto g \cdot x$ is continuous.
2. $g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x$ for all $g_1, g_2 \in G$ and $x \in X$.

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3. $e \cdot x = x$ for all $x \in X$, where e denotes the identity element of G.

It follows that G acts by homeomorphisms of X. Consider the topological spaces E^n (Euclidean n-space), $S^n \subset E^{n+1}$ (n-sphere), and $\mathcal{P}_n \subset E^2$ (regular n-sided polygon). Consider the groups D_n (dihedral group of order $2n$) and O_n ($n \times n$ orthogonal group). We then have the following transformation groups:

- (E^n, O_n).
- (S^n, O_{n+1}).
- $(\mathcal{P}_n, D_n)$.

The Fundamental Group

We want to know how to describe topological invariants associated with a transformation group (X, G). In order to do so, we need to understand how paths in X are affected by the action of G. Ultimately we will define equivalence classes of paths in X, taking into account the action of G, and a binary operation on the set of all such equivalence classes [1, Sec. 3].

Definition 2. Let (X, G) be a transformation group and x_0 be a point in X. Let I denote the interval $[0, 1]$ in the set of real numbers. Given $g \in G$, a *path of order g with base-point* x_0 is a continuous map $f : I \to X$ such that $f(0) = x_0$ and $f(1) = g \cdot x_0$.

All paths in X under consideration will have the same base-point but the order of the paths can vary. The *composition rule* for such paths is defined as follows.

Definition 3. Consider paths f_1 of order g_1 and f_2 of order g_2. We define the *composition* path $f_1 + g_1 f_2$ of order $g_1 g_2$ by

$$\begin{array}{lcll} (f_1 + g_1 f_2)(t) & = & f_1(2t) & \text{if } 0 \le t \le 1/2 \\ (f_1 + g_1 f_2)(t) & = & g_1 f_2(2t-1) & \text{if } 1/2 \le t \le 1. \end{array}$$

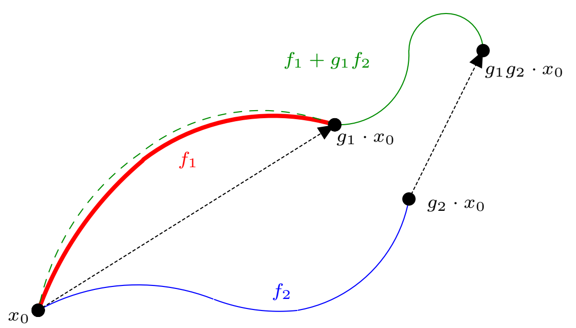


Figure 1: Composition of paths f_1 of order g_1 and f_2 of order g_2

Note that $(f_1 + g_1 f_2)(0) = f_1(0) = x_0$, $(f_1 + g_1 f_2)(1/2) = f_1(1) = g_1 f_2(0) = g_1 \cdot x_0$ and $(f_1 + g_1 f_2)(1) = g_1 f_2(1) = g_1 g_2 \cdot x_0$. Figure 1 illustrates the operation of path composition.

Definition 4. Let f_0 and f_1 be paths in X of the same order g. A *homotopy* from f_0 to f_1 is a continuous function $F : I \times I \to X$ such that for all $t, s \in I$, $F(t, 0) = f_0(t)$, $F(t, 1) = f_1(t)$, $F(0, s) = x_0$, and $F(1, s) = g \cdot x_0$.

If there exists a homotopy from f_0 to f_1, we say that f_0 and f_1 are *homotopic* and write $f_1 \asymp_{(g,x_0)} f_2$. The usage of this term is justified as follows.

Proposition 5. *The relation $\asymp_{(g,x_0)}$ is an equivalence relation on the set of all paths in X of order g.*

Proof. We need to show that the relation $\asymp_{(g,x_0)}$ is reflexive, symmetric, and transitive.

Let f be a path of order g. Consider the map F defined by homotopy $F(t, s) = f(t)$ for all $t, s \in I$. We have $F(0, s) = x_0$, $F(1, s) = g \cdot x_0$ and $F(t, 0) = F(t, 1) = f(t)$. Thus F is a homotopy from f to f, so that $f \asymp_{(g,x_0)} f$, meaning that the relation is reflexive.

If $f_0 \asymp_{(g,x_0)} f_1$ there exists a homotopy F from f_0 to f_1 as defined above. Consider the map $\overline{F}$ defined by $\overline{F}(t, s) = F(t, 1 - s)$. Then we have

$$\begin{array}{lclcl} \overline{F}(t, 0) & = & F(t, 1) & = & f_1(t), \\ \overline{F}(t, 1) & = & F(t, 0) & = & f_0(t), \\ \overline{F}(0, s) & = & x_0, & & \\ \overline{F}(1, s) & = & g \cdot x_0. & & \end{array}$$

Therefore $\overline{F}$ is a homotopy from f_1 to f_0, so the relation is symmetric.

If $f_0 \asymp_{(g,x_0)} f_1$ and $f_1 \asymp_{(g,x_0)} f_2$ then there exists a homotopy F_1 from f_0 to f_1 and a homotopy F_2 from f_1 to f_2. Consider the map F defined by

$$F(t, s) = \begin{cases} F_1(t, 2s) & \text{if } 0 \le s \le 1/2, \\ F_2(t, 2s-1) & \text{if } 1/2 \le s \le 1. \end{cases}$$

Note that $F(t, 1/2) = F_1(t, 1) = F_2(t, 0) = f_2(t)$, so that the homotopy is well-defined and continuous. Then

$$\begin{array}{lclcl} F(t, 0) & = & F_1(t, 0) & = & f_0(t), \\ F(t, 1) & = & F_2(t, 1) & = & f_2(t), \\ F(0, s) & = & F_1(0, s) = F_2(0, s) & = & x_0, \\ F(1, s) & = & F_1(1, s) = F_2(1, s) & = & g \cdot x_0. \end{array}$$

Therefore F is a homotopy from f_0 to f_2, so the relation is transitive. $\square$

We use this equivalence relation to define *homotopy classes* of paths in X with the same order. We denote by $[f; g]$ the equivalence class of a path f of order g. We define a binary operation $\star$ on homotopy classes based on the composition rule described in Definition 3:

$$[f_1; g_1] \star [f_2; g_2] = [f_1 + g_1 f_2; g_1 g_2].$$

It is essential to know that this operation is well-defined on the set of homotopy classes, that is, it depends only on the homotopy classes of f_1 and f_2.

The set of all equivalence classes $[f;g]$ is called the *fundamental group* of the transformation group (X,G) with base-point x_0 and will be denoted by $\pi_1(X,x_0,G)$.

Proposition 6. *The set $\pi_1(X,x_0,G)$ with the binary operation $\star$ is a group.*

Proof. If e is the identity element of the group G and $\mathbf{1}$ denotes the constant map $\mathbf{1}\colon I\to\{x_0\}$, then $[\mathbf{1};e]$ is the identity element of $\pi_1(X,x_0,G)$ since

$$[f;g]\star[\mathbf{1};e]=[f+g\mathbf{1};ge]=[f+g\cdot x_0;g]=[f;g]$$

and $[\mathbf{1};e]\star[f;g]=[\mathbf{1}+ef;eg]=[x_0+f;g]=[f;g]$.

Define $\overline{f}(t)=f(1-t)$. Then

$$\begin{aligned}
[f;g]\star\left[g^{-1}\overline{f};g^{-1}\right] &= [f+gg^{-1}\overline{f};gg^{-1}]\\
&= [f+\overline{f};e]\\
&= [\mathbf{1};e]\\
\left[g^{-1}\overline{f};g^{-1}\right]\star[f;g] &= [g^{-1}\overline{f}+g^{-1}f;g^{-1}g]\\
&= [g^{-1}(\overline{f}+f);g^{-1}g]\\
&= [g^{-1}(g\mathbf{1});e]\\
&= [\mathbf{1};e]
\end{aligned}$$

Thus the inverse element $[f;g]^{-1}=\left[g^{-1}\overline{f};g^{-1}\right]$ exists.

For associativity, we are going to use the fact that the operation $\star$ is well-defined, and prove this for an element of each equivalence class. Suppose $f_1\in[f_1;g_1]$, $f_2\in[f_2;g_2], and\ f_3\in[f_3;g_3]$. Then

$$(f_1+g_1f_2)+g_1g_2f_3=\begin{cases} f_1(4t) & 0\le t\le 1/4\\ g_1f_2(4t-1) & 1/4\le t\le 1/2\\ g_1g_2f_3(2t-1) & 1/2\le t\le 1,\end{cases}$$

$$f_1+(g_1f_2+g_1g_2f_3)=\begin{cases} f_1(2t) & 0\le t\le 1/2\\ g_1f_2(4t-2) & 1/2\le t\le 3/4\\ g_1g_2f_3(4t-3) & 3/4\le t\le 1.\end{cases}$$

Referring to Figure 2, if we want to define a homotopy between the paths described above it is necessary to delimit Regions I, II and III.

In Region I we have

$$0\ \le\ t\ \le\ \frac{s+1}{4},$$

so that

$$0\ \le\ \frac{4t}{s+1}\ \le\ 1.$$

In Region II we have

$$\frac{s+1}{4}\ \le\ t\ \le\ \frac{s+2}{4},$$

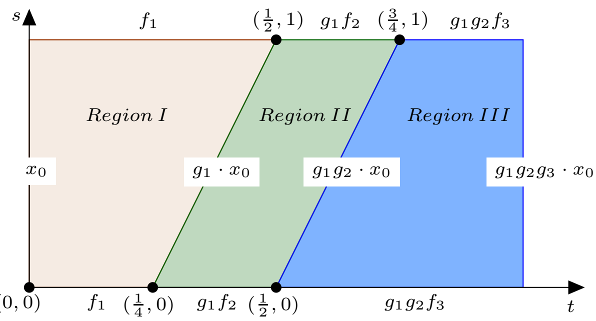


Figure 2: Homotopy between compositions of paths $\mathbf{f_1}$, $\mathbf{f_2}$, and $\mathbf{f_3}$.

so that

$$0\ \le\ 4t-s-1\ \le\ 1.$$

Finally, in Region III we have

$$\frac{s+2}{4}\ \le\ t\ \le\ 1,$$

so that

$$0\ \le\ \frac{4t-s-2}{2-s}\ \le\ 1.$$

We now consider the homotopy

$$F(t,s)=\begin{cases} f_1\left(\frac{4t}{s+1}\right) & 0\le t\le\frac{s+1}{4}\\ g_1f_2(4t-s-1) & \frac{s+1}{4}\le t\le\frac{s+2}{4}\\ g_1g_2f_3\left(\frac{4t-s-2}{2-s}\right) & \frac{s+2}{4}\le t\le 1.\end{cases}$$

Note that $F(0,s)=f_1(0)=x_0$ and $F(1,s)=g_1g_2f_3(1)=g_1g_2g_3x_0$, since

$$\begin{aligned}
F(t,0) &= \begin{cases} f_1(4t) & 0\le t\le 1/4\\ g_1f_2(4t-1) & 1/4\le t\le 1/2\\ g_1g_2f_3(2t-1) & 1/2\le t\le 1\end{cases}\\
&= ((f_1+g_1f_2)+g_1g_2f_3)(t),\\
F(t,1) &= \begin{cases} f_1(2t) & 0\le t\le 1/2\\ g_1f_2(4t-2) & 1/2\le t\le 3/4\\ g_1g_2f_3(4t-3) & 3/4\le t\le 1\end{cases}\\
&= (f_1+(g_1f_2+g_1g_2f_3))(t).
\end{aligned}$$

This proves associativity. $\square$

Consider an equivalence class $[f;e]$ which is a homotopy class of a path f of order the identity element e. Since e is the identity transformation of X, we have $f(0)=f(1)=x_0$, so that $[f;e]$ is a homotopy class of *loops* with base-point x_0. All such homotopy classes of loops form the ordinary *fundamental group of X with base-point* x_0. We denoted this group by $\pi_1(X,x_0)$ and note that it is a subgroup of $\pi_1(X,x_0,G)$. We shall denote $[\lambda;e]\in\pi_1(X,x_0)$ simply by $[\lambda]$.

Topological Properties

We will restrict our attention to path-connected spaces, so that the role of the base-point x_0 is inconsequential. Rhodes proves in his first theorem [1] that if ρ is a (continuous) path in X from x_0 to x_1, then the map

$$\begin{aligned} \rho_*\colon \pi_1(X,x_0,G) &\longrightarrow \pi_1(X,x_1,G) \\ [f;g] &\longmapsto [\overline{\rho}+f+g\rho] \end{aligned}$$

is an isomorphism. More generally, a pair of mappings

$$(\varphi,\psi)\colon (X,G) \longrightarrow (Y,H)$$

in which $\varphi : X \to Y$ is a continuous map and $\psi : G \to H$ is a group homomorphism induces a homomorphism $(\varphi,\psi)_*$ of fundamental groups [1, Sec. 5]:

$$\begin{aligned} (\varphi,\psi)_*\colon \pi_1(X,x_0,G) &\longrightarrow \pi_1(Y,y_0,H) \\ [f;g] &\longmapsto [\varphi(f);\psi(g)]. \end{aligned}$$

We say that the transformation groups (X,G) and (Y,H) have the same *homotopy type* if there exist pairs of mappings

$$\begin{aligned} (\varphi,\psi)\colon\ (X,G) &\longrightarrow (Y,H) \\ (\varphi',\psi')\colon (Y,H) &\longrightarrow (X,G) \end{aligned}$$

such that $\varphi'\varphi$ and $\varphi\varphi'$ are homotopic to the identity maps of X and Y, respectively, and ψ and ψ' are isomorphisms. Rhodes proves that the fundamental group of a transformation group is an invariant of the homotopy type of its transformation group [1, Sec. 5].

Relationship between $\pi_1(X,x_0,G)$ and $\pi_1(X,x_0)$

Let $[\lambda] \in \pi_1(X,x_0)$ and $[f;g] \in \pi_1(X,x_0,G)$. Observe that

$$\begin{aligned} [f;g]\star[\lambda;e]\star[g^{-1}\overline{f};g^{-1}] &= ([f;g]\star[\lambda;e])\star[g^{-1}\overline{f};g^{-1}] \\ &= [f+g\lambda;ge]\star[g^{-1}\overline{f};g^{-1}] \\ &= [f+g\lambda+gg^{-1}\overline{f};gg^{-1}] \\ &= [f+g\lambda+\overline{f};e]. \end{aligned}$$

This establishes that $\pi_1(X,x_0)$ is a *normal* subgroup of $\pi_1(X,x_0,G)$.

Let us consider the inclusion map

$$i\colon \quad \pi_1(X,x_0) \hookrightarrow \pi_1(X,x_0,G)$$

such that $i([\lambda]) = [\lambda] = [\lambda;e]$, which is an injective homomorphism (*monomorphism*). Let

$$p\colon \quad \pi_1(X,x_0,G) \twoheadrightarrow G$$

be the map $p([f;g]) = g$, which is a surjective homomorphism (*epimorphism*).

Definition 7. [4, Chap. 7] An *exact sequence* is a sequence of objects (e.g. vector spaces, groups) and morphisms between them (e.g. linear maps, homomorphisms) such that the image of each morphism in the sequence is equal to the kernel of the next morphism in the sequence.

Note that $\mathrm{Im}(i) = \ker(p) = \pi_1(X,x_0)$ so that we have an exact sequence

$$\pi_1(X,x_0) \overset{i}{\hookrightarrow} \pi_1(X,x_0,G) \overset{p}{\twoheadrightarrow} G \quad (1)$$

in which i is a monomorphism and p is an epimorphism. Such an exact sequence is known as a *short exact sequence*. It follows [2, Chap. 3] that the *quotient group*

$$\pi_1(X,x_0,G)/\pi_1(X,x_0)$$

is isomorphic to G.

In order to obtain a more explicit relationship between the fundamental groups $\pi_1(X,x_0,G)$ and $\pi_1(X,x_0)$, we need to be able to relate loops in X based at x_0 with general paths of order g [1, Sec. 9].

Preferred Paths

Definition 8. The transformation group (X,G) admits a family of *preferred paths* $\{k_g \mid g \in G\}$ at x_0 if it is possible to associate to each $g \in G$ a path k_g in X in such a way that:

1. For all $g \in G$, $k_g(0) = g\cdot x_0$ and $k_g(1) = x_0$.
2. The path k_e associated with the identity element $e \in G$ is constant.
3. For all $g_1, g_2 \in G$ the path $k_{g_1g_2}$ is homotopic to $g_1k_{g_2} + k_{g_1}$.

If G is a *topological group*, then G acts on itself by homeomorphisms via translations. A family of preferred paths $\{h_g \mid g \in G\}$ at the identity element $e \in G$ then induces a family $\{k_g \mid g \in G\}$ of preferred paths at $x_0 \in X$ as follows: $k_g(t) = h_g(t)\cdot x_0$, $\forall t \in I$.[1, Sec. 9]

The existence of a family of preferred paths leads to a more explicit relationship between both fundamental groups, $\pi_1(X,x_0)$ and $\pi_1(X,x_0,G)$. For each $g \in G$, we have an automorphism K_g of $\pi_1(X,x_0)$ defined by

$$\begin{aligned} K_g\colon \pi_1(X,x_0) &\longrightarrow \pi_1(X,x_0) \\ [\lambda] &\longmapsto [\overline{k_g}+g\lambda+k_g]. \end{aligned}$$

For $g_1, g_2 \in G$ we have

$$\begin{aligned} K_{g_1}(K_{g_2}([\lambda])) &= K_{g_1}([\overline{k_{g_2}}+g_2\lambda+k_{g_2}]) \\ &= [\overline{k_{g_1}}+g_1(\overline{k_{g_2}}+g_2\lambda+k_{g_2})+k_{g_1}] \\ &= [\overline{k_{g_1}}+g_1\overline{k_{g_2}}+g_1g_2\lambda+g_1k_{g_2}+k_{g_1}]. \end{aligned}$$

Since the composition rule $+$ is well-defined on homotopy classes of paths, we can take any representative of the equivalence class. Recalling that $g_1k_{g_2}+k_{g_1}$ is homotopic to $k_{g_1g_2}$, we have that $\overline{k_{g_1}}+g_1\overline{k_{g_2}}$ is homotopic

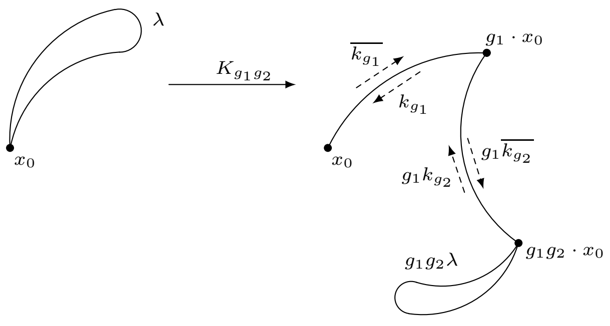


Figure 3: Automorphism induced by $\mathbf{k_{g_1}}, \mathbf{k_{g_2}}$

to $\overline{k_{g_1g_2}}$. Thus $\overline{k_{g_1}} + g_1\overline{k_{g_2}} + g_1g_2\lambda + g_1k_{g_2} + k_{g_1} \sim_{x_0} \overline{k_{g_1g_2}} + g_1g_2\lambda + k_{g_1g_2}$. Thus we have

$$\begin{aligned} K_{g_1}(K_{g_2}([\lambda])) &= [\overline{k_{g_1g_2}} + g_1g_2\lambda + k_{g_1g_2}] \\ &= K_{g_1g_2}([\lambda]), \end{aligned}$$

and so $K_{g_1} \circ K_{g_2} = K_{g_1g_2}$. This automorphism is illustrated in Figure 3.

Thus we see that the map $g \to K_g$ defines a homomorphism

$$K\colon \quad G \longrightarrow \mathrm{Aut}(\pi_1(X, x_0)) \tag{2}$$

from G into the group $\mathrm{Aut}(\pi_1(X, x_0))$ of automorphisms of $\pi_1(X, x_0)$ [1, Sec. 9].

Let us now consider the product set $\pi_1(X, x_0) \times G$ of all ordered pairs $([\lambda], g)$ where $[\lambda] \in \pi_1(X, x_0)$ and $g \in G$.

Definition 9. [2, Chap. 5] Given a group G that acts on a group H by group automorphism via $\varphi\colon G \to \mathrm{Aut}(H)$, the *semidirect product* group, denoted by $H \rtimes_\varphi G$ (or simply $H \rtimes G$) is the group whose underlying set is the product set $H \times G$, but whose group operation is defined by

$$(h_1, g_1)(h_2, g_2) = (h_1\varphi(g_1)(h_2), g_1g_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$.

Thus a family of preferred paths allows us to form the semidirect product group

$$\pi_1(X, x_0) \rtimes G$$

in which we have the group operation

$$([\lambda_1], g_1)([\lambda_2], g_2) = ([\lambda_1 + K_{g_1}(\lambda_2)], g_1g_2)$$

for any $([\lambda_1], g_1), ([\lambda_2], g_2) \in \pi_1(X, x_0) \times G$.

Rhodes's Theorem

We are going to use the semidirect group $\pi_1(X, x_0) \rtimes G$ in order to obtain an explicit relation between the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_0, G)$.

Theorem 10. *Suppose that (X, G) admits a family of preferred paths at x_0. Then the map*

$$\begin{aligned} \phi\colon \pi_1(X, x_0, G) &\longrightarrow \pi_1(X, x_0) \rtimes G \\ [f; g] &\longmapsto ([f + k_g], g) \end{aligned}$$

is an isomorphism. Moreover, if (G, G) admits a family of preferred paths at e, then for every transformation group (X, G), ϕ is an isomorphism.

Proof. Note that for $a = [f_1; g_1], b = [f_2; g_2]$:

$$\begin{aligned} \phi(a) \star \phi(b) &= ([f_1 + k_{g_1}]; g_1) \star ([f_2 + k_{g_2}]; g_2) \\ &= ([f_1 + k_{g_1} + K_{g_1}(f_2 + k_{g_2})], g_1g_2) \\ &= ([f_1 + k_{g_1} + \overline{k_{g_1}} + g_1(f_2 + k_{g_2}) + k_{g_1}], g_1g_2) \\ &= ([f_1 + \mathbf{1} + g_1f_2 + g_1k_{g_2} + k_{g_1}], g_1g_2) \\ &= ([f_1 + g_1f_2 + k_{g_1g_2}], g_1g_2) \end{aligned}$$

Since $g_1k_{g_2} + k_1$ is homotopic to $k_{g_1g_2}$ (Definition 8). It is also true that:

$$\begin{aligned} \phi(a \star b) &= \phi([f_1 + g_1f_2, g_1g_2]) \\ &= ([f_1 + g_1f_2 + k_{g_1g_2}], g_1g_2) \end{aligned}$$

Thus, the map ϕ is an homomorphism.

Let $[f_1; g_1], [f_2; g_2] \in \pi_1(X, x_0, G)$ such that $[f_1; g_1] \neq [f_2; g_2]$. If $g_1 = g_2 = g$, then f_1 and f_2 are not homotopy equivalent, and so $f_1 + k_g$ and $f_2 + k_g$ cannot be homotopy equivalent. Hence $([f_1 + k_{g_1}], g_1) \neq ([f_2 + k_{g_2}], g_2)$. Thus ϕ is injective.

Consider the map

$$\begin{aligned} s\colon \quad G &\longrightarrow \pi_1(X, x_0, G) \\ g &\longmapsto [\overline{k_g}; g]. \end{aligned}$$

Recall that $\overline{k_{g_1}} + g_1\overline{k_{g_2}}$ is homotopic to $\overline{k_{g_1g_2}}$, then:

$$\begin{aligned} s(g_1)s(g_2) &= [\overline{k_{g_1}}; g_1] \star [\overline{k_{g_2}}; g_2] \\ &= [\overline{k_{g_1}} + g_1\overline{k_{g_2}}; g_1g_2] \\ &= [\overline{k_{g_1g_2}}; g_1g_2] \\ &= s(g_1g_2). \end{aligned}$$

Thus s is a homomorphism. Consider the short exact sequence described in Equation 1. Note that

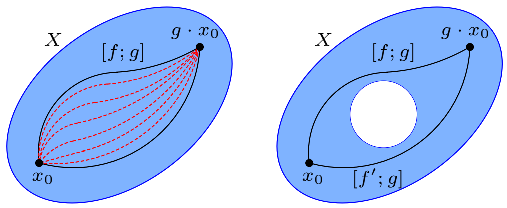
$$p \circ s(g) = p([\overline{k_g}; g]) = g.$$

Thus $p \circ s = \mathrm{id}_G$, the identity map of G. Recall that $\ker(p) = \pi_1(X, x_0)$. Then

$$\begin{aligned} p(s \circ p([f; g]) \star [f; g]^{-1}) &= p(s \circ p([f; g]))p([f; g]^{-1}) \\ &= p([f; g])p([f; g]^{-1}) \\ &= e. \end{aligned}$$

Therefore $s \circ p([f; g]) \star [f; g]^{-1} \in \ker(p)$. Finally,

$$\begin{aligned} s \circ p([f; g]) \star [f; g]^{-1} &= s(g) \star [g^{-1}\overline{f}; g^{-1}] \\ &= [\overline{k_g}; g] \star [g^{-1}\overline{f}; g^{-1}] \\ &= [\overline{k_g} + gg^{-1}\overline{f}; gg^{-1}] \\ &= [\overline{k_g} + \overline{f}; e]. \end{aligned}$$



(a) Simply connected (b) Non-simply connected

Figure 4: Representation of equivalence classes for paths of order g

Since $[\overline{k_g} + \overline{f}; e] \in \pi_1(X, x_0)$, we have $[f + k_g; e] \in \pi_1(X, x_0)$. This proves surjectivity, since any loop λ is homotopic to certain loop of the form $f + k_g$.

As was pointed out, the condition for (G, G) to admit a family of preferred paths at e is equivalent to the condition that every transformation group (X, G) admits a family of preferred paths.

$\square$

Remark 11. The map s in the proof of Theorem 10 is known as a *splitting map* for the exact sequence 1. In general, the existence of a splitting map establishes an isomorphism with a semidirect product.

We now mention some direct corollaries of Theorem 10.

For a *simply connected* space X we have that $\pi_1(X, x_0)$ is the trivial one-element group, and any two paths with the same order are homotopic. Thus the set of paths of order g make up a *unique* equivalence class $[f; g]$. Figure 4(b) shows how a non-simply connected space can have more than one equivalence class for paths of order g.

In the simply connected case, choosing any path from $g \cdot x_0$ to x_0 provides a family of preferred paths, and the semidirect product in Theorem 10 is isomorphic to G.

Corollary 12. *If X is simply connected, then $\pi_1(X, x_0, G) \cong G$.*

If x_0 is a *fixed point* of (X, G), that is, $g \cdot x_0 = x_0$ for all $g \in G$, then the constant map **1** alone provides a family of preferred paths at x_0.

Corollary 13. *If $x_0 \in X$ is a fixed point of (X, G), then $\pi_1(X, x_0, G) \cong \pi_1(X, x_0) \rtimes G$ where the action of G on $\pi_1(X, x_0)$ is given by*

$$\begin{aligned} K_g\colon \pi(X, x_0) &\longrightarrow \pi(X, x_0) \\ [\lambda] &\longmapsto [g\lambda]. \end{aligned}$$

Examples

We now look at some examples.

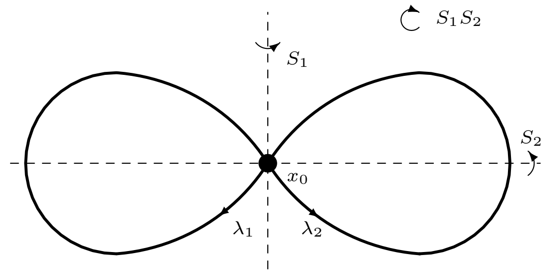


Figure 5: Figure eight space: $\mathfrak{E}$

Example 1. *Consider the transformation groups (E^n, O_n) and $(S^m, O_{m+1}), m > 1$ in which the orthogonal group acts on Euclidean space and the sphere, respectively. Since E^n and $S^m, m > 1$ are simply connected topological spaces, we have $\pi_1(E^n, O_n) \cong O_n$ and $\pi_1(S^m, O_{m+1}) \cong O_{m+1}, m > 1$.*

Example 2. *Consider the unit disk $\mathcal{D} = \{(x, y) \mid d((x, y), (0, 0)) \leq 1\} \subseteq E^2$. Since $\mathcal{D}$ is simply connected, $\pi_1(\mathcal{D}, O_2) \cong O_2$.*

Example 3. *Consider the topological group $(\mathbb{R}, +)$ acting on itself by translation. Then $\pi_1(\mathbb{R}, 0, \mathbb{R}) \cong \mathbb{R}$. More generally, we can replace $\mathbb{R}$ by any simply connected topological group and obtain the same result.*

We now turn to non-simply connected spaces.

Example 4. *Consider the figure-eight space $\mathfrak{E}$ depicted in Figure 5. The fundamental group is the* free group *generated by the loops λ_1, λ_2. This topological space is path-connected and non-simply connected. as said before, we know that $\pi_1(\mathfrak{E}, x_0) \cong \langle\lambda_1, \lambda_2\rangle \cong \mathbb{F}_2$. The group acting over $\mathfrak{E}$ is the one generated by the reflections S_1 and S_2. Note that this describes the dihedral group D_2 also known as* Klein four-group, *which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The action of D_2 over $\mathfrak{E}$ fixes the point x_0, thus by Corollary 13 we can conclude that:*

$$\pi_1(\mathfrak{E}, D_2) \cong \mathbb{F}_2 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

In which the automorphism of $\mathbb{F}_2$ induced by D_2 is given by:

$$\begin{array}{rlll} K_0\colon & \mathbb{F}_2 \longrightarrow \mathbb{F}_2 & K_{S_2}\colon & \mathbb{F}_2 \longrightarrow \mathbb{F}_2 \\ & \lambda_1 \rightleftarrows \lambda_1 & & \lambda_1 \rightleftarrows \overline{\lambda_1} \\ K_{S_1}\colon & \mathbb{F}_2 \longrightarrow \mathbb{F}_2 & K_{S_1S_2}\colon & \mathbb{F}_2 \longrightarrow \mathbb{F}_2 \\ & \lambda_1 \rightleftarrows \lambda_2 & & \lambda_1 \rightleftarrows \overline{\lambda_2} \end{array}$$

Example 5. *Consider the topological space described in Figure 6. It is basically the regular polygon of n sides with extra sides joining all vertices with the point x_0 (center of rotation), let's call it $\tilde{\mathcal{P}}_n$. If we let the dihedral group D_n to act on $\tilde{\mathcal{P}}_n$, it is clear that the point x_0 is fixed under the action; thus, we can calculate the fundamental group of the transformation group $(\tilde{\mathcal{P}}_n, D_n)$ as follows:*

$$\pi_1(\tilde{\mathcal{P}}_n, x_0, D_n) \cong \mathbb{F}_n \rtimes D_n$$

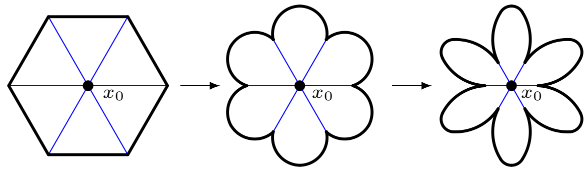


Figure 6: Descomposition of $\tilde{\mathcal{P}}_n$ in its generators

The automorphism of $\mathbb{F}_n$ induced by the elements of D_n are similar to the ones described in Example 4, for there are elements in D_n that map λ_i onto $\lambda_i, \overline{\lambda_i}, \lambda_j$ or $\overline{\lambda_j}$ for $j \neq i$.

The topological group SO_2

Consider the special orthogonal group SO_2, which is homeomorphic to the circle S^1, and thus acts continuously on S^1 (by rotations). Let $e = R_0 \in SO_2$, then we have the path $f\colon I \to SO_2$ such that $f(t) = R_{(1-t)\theta}$.

However, this does *not* define a family of preferred paths in S^1. If $f_{\theta_1}(t) = R_{(1-t)\theta_1}$, $f_{\theta_2}(t) = R_{(1-t)\theta_2}$ and $f_{\theta_{1,2}}(t) = R_{(1-t)(\theta_1+\theta_2)}$, it is possible to prove that indeed $f_{\theta_{1,2}}$ is homotopic to $R_{\theta_1} f_{\theta_2} + f_{\theta_1}$. Consider the case when $\theta_1 = \theta_2 = \pi$. The result is the path from R_0 to $R_{2\pi}$ and it is also true that $R_0 = R_{2\pi}$. However, the constant path f_0 is not homotopic to the path $f_{2\pi}$. Indeed no family of preferred paths exists, so that Rhodes's theorem does not apply to the transformation group (S^1, SO_2).

Actions on $\mathbf{S^1}$

Consider the cyclic group of order n acting on S^1. Let $R_{\frac{1}{n}} \in \mathbb{Z}_n$ be the counterclockwise rotation of $\frac{2\pi}{n}$ radians and $f_{\frac{1}{n}}$ be the path from x_0 to $R_{\frac{1}{n}} \cdot x_0$.

Theorem 14. *$\pi_1(S^1, \mathbb{Z}_n) \cong \mathbb{Z}$.*

Proof. Consider the element $[f_{\frac{1}{n}}; R_{\frac{1}{n}}] \in \pi_1(S^1, \mathbb{Z}_n)$. Then:

$$
\begin{aligned}
[f_{\frac{1}{n}}; R_{\frac{1}{n}}]^2 &= [f_{\frac{1}{n}} + R_{\frac{1}{n}} f_{\frac{1}{n}}; R^2_{\frac{1}{n}}]^2 \\
&= [f_{\frac{2}{n}}; R_{\frac{2}{n}}] \\
[f_{\frac{1}{n}}; R_{\frac{1}{n}}]^3 &= [f_{\frac{2}{n}}; R_{\frac{2}{n}}] \star [f_{\frac{1}{n}}; R_{\frac{1}{n}}] \\
&= [f_{\frac{2}{n}} + R_{\frac{2}{n}} f_{\frac{1}{n}}; R_{\frac{2}{n}} R_{\frac{1}{n}}] \\
&= [f_{\frac{3}{n}}; R_{\frac{3}{n}}] \\
\vdots \quad & \vdots \quad \vdots \\
[f_{\frac{1}{n}}; R_{\frac{1}{n}}]^m &= [f_{\frac{m-1}{n}}; R_{\frac{m-1}{n}}] \star [f_{\frac{1}{n}}; R_{\frac{1}{n}}] \\
&= [f_{\frac{m-1}{n}} + R_{\frac{m-1}{n}} f_{\frac{1}{n}}; R_{\frac{m-1}{n}} R_{\frac{1}{n}}] \\
&= [f_{\frac{m}{n}}; R_{\frac{m}{n}}]
\end{aligned}
$$

Figure 7 represents the actions of different rotations $R_{\frac{m}{n}}$ over the path $f_{\frac{1}{n}}$.

Recall from Proposition 6 that the inverse element of $[f_{\frac{1}{n}}; R_{\frac{1}{n}}]$ is $[R^{-1}_{\frac{1}{n}} \overline{f_{\frac{1}{n}}}; R^{-1}_{\frac{1}{n}}]$. The rotation $R^{-1}_{\frac{1}{n}} = R_{-\frac{1}{n}}$

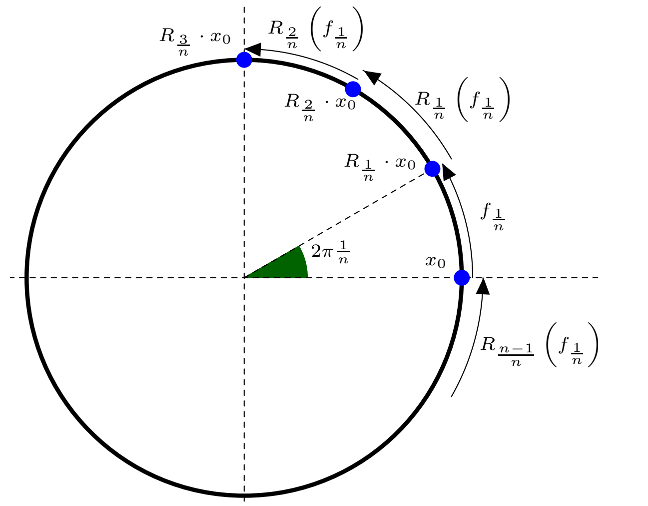


Figure 7: Representation of positive powers of $[\mathbf{f}_{\frac{1}{n}}; \mathbf{R}_{\frac{1}{n}}]$

is the clockwise rotation of $\frac{2\pi}{n}$ radians, and the path $\overline{f_{\frac{1}{n}}}$ goes from $R_{\frac{1}{n}} \cdot x_0$ to x_0 (clockwise direction); therefore, the path $R^{-1}_{\frac{1}{n}} \overline{f_{\frac{1}{n}}} = f_{-\frac{1}{n}}$ goes from x_0 to $R_{-\frac{1}{n}} \cdot x_0$ in a clockwise direction. Using $[f_{\frac{1}{n}}; R_{\frac{1}{n}}]^{-1} = [f_{-\frac{1}{n}}; R_{-\frac{1}{n}}]$ we can conclude that:

$$
\begin{aligned}
[f_{-\frac{1}{n}}; R_{-\frac{1}{n}}]^2 &= [f_{-\frac{1}{n}} + R_{-\frac{1}{n}} f_{-\frac{1}{n}}; R^2_{-\frac{1}{n}}] \\
&= [f_{-\frac{2}{n}}; R_{-\frac{2}{n}}] \\
[f_{-\frac{1}{n}}; R_{-\frac{1}{n}}]^3 &= [f_{-\frac{2}{n}}; R_{-\frac{2}{n}}] \star [f_{-\frac{1}{n}}; R_{-\frac{1}{n}}] \\
&= [f_{-\frac{2}{n}} + R_{-\frac{2}{n}} f_{-\frac{1}{n}}; R_{-\frac{2}{n}} R_{-\frac{1}{n}}] \\
&= [f_{-\frac{3}{n}}; R_{-\frac{3}{n}}] \\
\vdots \quad & \vdots \quad \vdots \\
[f_{-\frac{1}{n}}; R_{-\frac{1}{n}}]^m &= [f_{-\frac{m-1}{n}}; R_{-\frac{m-1}{n}}] \star [f_{-\frac{1}{n}}; R_{-\frac{1}{n}}] \\
&= [f_{-\frac{m-1}{n}} + R_{-\frac{m-1}{n}} f_{-\frac{1}{n}}; R_{-\frac{m-1}{n}} R_{-\frac{1}{n}}] \\
&= [f_{-\frac{m}{n}}; R_{-\frac{m}{n}}]
\end{aligned}
$$

This clearly reflects an *additive* structure under the composition rule $\star$, the isomorphism with the integers is given by the mapping of the generator $[f_{\frac{1}{n}}; R_{\frac{1}{n}}] \in \pi_1(S^1)$ to the generator $1 \in \mathbb{Z}$. Since S^1 is path connected, it does not depend on the base-point chosen. Therefore $\pi_1(S^1, \mathbb{Z}_n) \cong \mathbb{Z}$. Figure 8 shows the actions of some clockwise rotations on the inverse element of the generator $[f_{\frac{1}{n}}; R_{\frac{1}{n}}]$. □

$\mathbf{S^1}$ acting on Topological Spaces

Suppose that S^1 acts on a topological space X. For $x \in X$, the *orbit* $S^1 \cdot x_0$ defines and homotopy class on $\pi_1(X, x_0)$. Recall that $\pi_1(S^1) \cong \mathbb{Z}$, then there exists an homomorphism: [5, Sec. 4]

$$
\begin{aligned}
\alpha\colon \pi_1(S^1) &\longrightarrow \pi_1(X, x_0) \\
n &\longmapsto \alpha(n)
\end{aligned}
$$

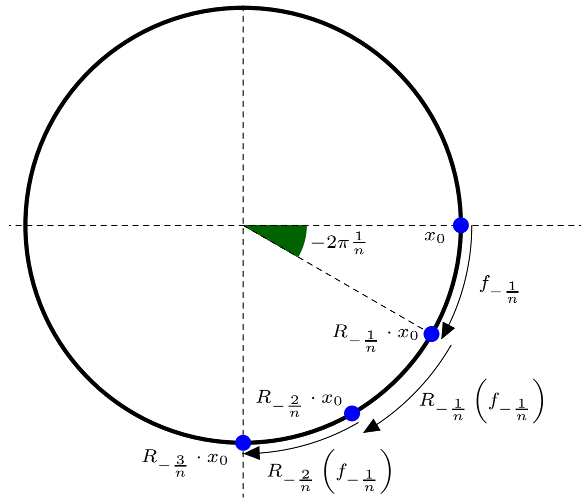


Figure 8: Representation of negative powers of $[\mathbf{f}_{\frac{1}{n}}; \mathbf{R}_{\frac{1}{n}}]$

Where $\alpha(n)$ is the path from x_0 to itself following the S^1-orbit n times. Note that there is enough to consider what is the image of 1 under the homomorphism α, since $\mathbb{Z} = \langle 1 \rangle$ and $\alpha(n)+\alpha(m) = \alpha(n+m)$; therefore, $\alpha(\pi_1(S^1)) = \langle \alpha(1) \rangle$.

Consider the $\mathbb{Z}$-sets: $\pi_1(X, x_0)$ and $\mathbb{R}$. Let $\mathbb{Z}$ be acting on $\pi_1(X, x_0)$ via the homomorphism α and $\mathbb{Z}$ be acting on $\mathbb{R}$ by *translation* as follows:

$$\begin{aligned} \mathbb{Z} \times \mathbb{R} &\longrightarrow \mathbb{R} & \mathbb{Z} \times \pi_1(X, x_0) &\longrightarrow \pi_1(X, x_0) \\ (n, r) &\longmapsto n + r & (n, \lambda) &\longmapsto \alpha(n) + \lambda \end{aligned}$$

Now, consider the *fiber product* over $\mathbb{Z}$ [6, Appendix III]:

$$\pi_1(X, x_0) \times_{\mathbb{Z}} \mathbb{R} = (\pi_1(X, x_0) \times \mathbb{R})/\mathbb{Z}$$

This set is the *quotient group* of the group $\pi_1(X, x_0) \times \mathbb{R}$ under the equivalence relation $(\lambda, r + n) \sim (\alpha(n) + \lambda, r)$.

Theorem 15. $\pi_1(X, x_0, S^1) \cong \pi_1(X, x_0) \times_{\mathbb{Z}} \mathbb{R}$

Proof. Let $x \in X$ and $r \in \mathbb{R}$, then $r_x \in X$ be the path from x to $r \cdot x \in S^1$-orbit. Consider the following map:

$$\begin{aligned} \rho\colon \pi_1(X, x_0) \times \mathbb{R} &\longrightarrow \pi_1(X, x_0, S^1) \\ (\lambda, r) &\longmapsto [\lambda + r_{x_0}; r \mod 1] \end{aligned}$$

Recall that $r \mod 1 \in S^1$ since $S^1 \cong \mathbb{R}/\mathbb{Z}$. [7] Note that ρ is onto since $\lambda + r_{x_0}$ is a path of order $g \equiv r \mod 1$. It is clear that $r \equiv r + n \mod 1$ for $n \in \mathbb{Z}$; moreover, $(\alpha(n)+\lambda)+r_{x_0} \asymp_{(r \mod 1, x_0)} \lambda+(r+n)_{x_0}$ since $\alpha(n) = n_{x_0}$ (because they follow the same orbit with the base-point x_0). Thus, the map ρ is not one-to-one. Note that if $(\lambda, r + n) \sim (\alpha(n) + \lambda, r)$, then $\rho_*\colon \pi_1(X, x_0) \times_{\mathbb{Z}} \mathbb{R} \to \pi_1(X, x_0, S^1)$ is an isomorphism. □

Corollary 16. $\pi_1(S^1, SO_2) \cong \mathbb{R}$

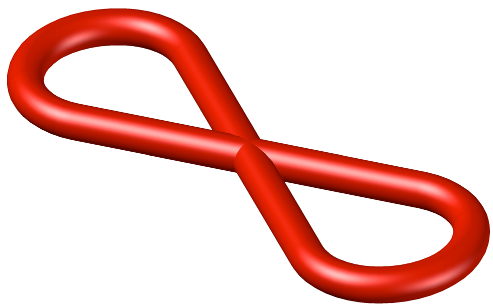


Figure 9: The 3D figure eight as a cartesian product $\mathfrak{E} \times \mathbf{S}^1$

Proof. From the previous theorem consider $X = S^1$, then $\pi_1(S^1, S^1) \cong \pi_1(S^1) \times_{\mathbb{Z}} \mathbb{R} \cong \mathbb{Z} \times \mathbb{R}/\mathbb{Z} \cong \mathbb{R}$. As $S^1 \cong SO_2$, we get the isomorphism required. □

Product Spaces

Using the topological spaces and groups studied so far, it is possible to construct new transformation groups using direct products. For instance, let $(X, G), (Y, H)$ be transformation groups, then every pair of action $(g, h) \in G \times H$ gives rise to an homeomorphism: [1, Sec. 10]

$$\begin{aligned} (g, h)\colon \quad X \times Y &\longrightarrow X \times Y \\ (x, y) &\longmapsto (g \cdot x, h \cdot y) \\ (x_1, y_1)(x_2, y_2) &\longmapsto (g \cdot x_1 x_2, h \cdot y_1 y_2) \end{aligned}$$

Proposition 17. *The mapping*

$$\begin{aligned} \pi_1(X, x_0, G) \times \pi_1(Y, y_0, H) &\longrightarrow \pi_1(X \times Y, (x_0, y_0), G \times H) \\ ([f_x; g], [f_y; h]) &\longmapsto [(f_x, f_y); (g, h)] \end{aligned}$$

is an isomorphism.

Proof. To prove that the mapping is a homomorphism is enough to say that the direct product is composed by a morphism in each coordinate. Note that, the projections:

$$\begin{aligned} p_1\colon f_*(\pi_1(X, x_0, G) \times \{e\}) &\longrightarrow \pi_1(X, x_0, G) \\ p_2\colon f_*(\{e\} \times \pi_1(Y, y_0, H)) &\longrightarrow \pi_1(Y, y_0, H) \end{aligned}$$

are both bijections; therefore the map is an isomorphism. □

Example 6. *Consider the group G acting on $\mathbb{R}$. By the Corollary 12 we know that $\pi_1(\mathbb{R}, G) \cong G$. Consider the cartesian product $G^n = \prod_{i=1}^n G$. Then we can use Proposition 17 to calculate the fundamental group of the trasnformation group $(\mathbb{R}^n, G \times \cdots \times G)$ as follows:*

$$\pi_1(\mathbb{R}^n, G^n) \cong \prod_{i=1}^{n} \pi_1(\mathbb{R}, G) \cong G^n$$

which is the desired result as of Corollary 12.

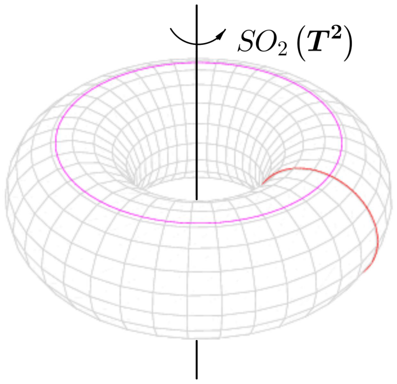


Figure 10: The torus $\mathbf{T^2}$ as a cartesian product $\mathbf{S^1}\times\mathbf{S^1}$

Example 7. *Consider the topological space described in Figure 9. It is a representation of the cartesian product of the figure eight described in Example 5 and the circle: $\mathfrak{E} \times S^1$. If we want the dihedral group acting over the 3D figure eight, it is possible to use a product space using a trivial group action over the circle as follows:*

$$\pi_1(\mathfrak{E}\times S^1, D_2) \cong \pi_1(\mathfrak{E}, D_2)\times\pi_1(S^1, \{e\}) \cong (\mathbb{F}_2 \rtimes D_2)\times\mathbb{Z}$$

Example 8. *Consider the torus $\mathbf{T^2} = S^1 \times S^1$. Let SO_2 act on $\mathbf{T^2}$ via rotations with respect to its axis of symmetry for rotations. The fundamental group of the transformation group $(\mathbf{T^2}, SO_2)$ can be calculated using a cartesian product as well, using the same trick as before of letting act a trivial group $\{e\}$ on one of the cartesian components of the torus:*

$$\pi_1(\mathbf{T^2}, SO_2) \cong \pi_1(S^1, SO_2) \times \pi_1(S^1, \{e\}) \cong \mathbb{R} \times \mathbb{Z}$$

Conclusions

The article written by Rhodes opened the door for a whole new category in algebraic topology. His ideas drew a new connection between the worlds of topology and abstract algebra. Although his results were not so applicable at first sight, mathematicians soon came to find them very useful. It is important to emphasize the results obtained when studied the topological group S^1. Recall that:

$$\begin{aligned} \pi_1(S^1, \mathbb{Z}_n) &\cong \mathbb{Z} \\ \pi_1(S^1, SO_2) &\cong \mathbb{R} \end{aligned}$$

Therefore, when a cyclic group acts on S^1 its fundamental group is a discrete group. In the other hand, when a topological group acts on S^1 its fundamental group is a topological group. What can be garnered from taking a deeper look at this type of work is that there are still some unanswered questions for relatively easy transformation groups. For example, we still do not know how to calculate the fundamental group of (S^1, O_2).

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Symbols

(X, G)	Transformation group
$\mathbf{1}$	Constant function $f(X) = x_0$
$[\lambda]$	Equivalence class of λ
$\pi_1(X, x_0)$	Fund. group of X with base point x_0
$[f; g]$	Equivalence class of paths of order g
$\pi_1(X, x_0, G)$	Fund. group of (X, G) with base point x_0
$\{k_g \mid g \in G\}$	Family of preferred paths
R_θ	Rotation of angle θ
E^n	Euclidean Space of dimension n
$\mathfrak{E}$	Figure-eight space
S^n	Sphere of dimension n
D_n	Dihedral group of order $2n$
O_n	Orthogonal group of dimension n
$\mathbb{F}_n$	Free group of n generators
$\mathbb{Z}_n$	Cyclic group of order n
$\cong$	Isomorphic to
$\sim_{x_0}$	Homotopic modulo x_0
$\asymp_{(g,x_0)}$	Paths of order g modulo x_0
$\trianglelefteq$	Normal subgroup of
$\rtimes$	Semidirect product
$\times_G$	Fiber product over G
$\langle a \rangle$	Group generated by a

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